Accretion & disks

The disks are everywhere

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Bondi-Hoyle-Lyttleton accretion theory

Bondi radius

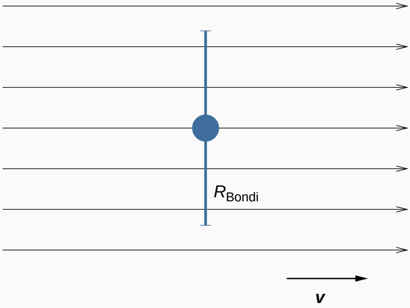
Let us assume that the medium (ISM, wind, ...) moves with velocity $\boldsymbol{v}$ with respect to the accreting object.

The object is able to accrete only particles with negative total energy,

$$E_{\mathrm{kin}} + E_{\mathrm{pot}} < 0.$$

Assuming spherically symmetric gravitaional field,

$$\frac{1}{2}\rho v^2 < \rho \frac{GM}{r}.$$



This gives condition for a limiting **Bondi radius**, from which a spherically symmetric object can accrete matter, as

$$R_{\mathrm{Bondi}} = \frac{2GM}{v^2}.$$

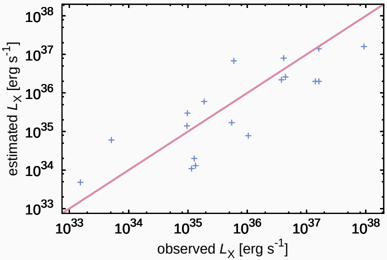
Mass-accretion rate from stellar wind in a binary

The total amount of accreted matter per unit of time (mass-accretion rate) is given by matter swept by a circle with radius R_{Bondi},

$$\dot{M}_{\mathrm{acc}} = \dot{M}\frac{R_{\mathrm{Bondi}}^2}{4D^2}.$$

Here D is the orbital separation. For a compact accreting object (white dwarf, neutron star, or a black hole) with radius R, the energy released by accretion (per unit of time) is

$$L_{\mathrm{X}} = \dot{M}_{\mathrm{acc}}\frac{GM}{R}.$$



(Krtička et al. 2019)

This process can explain X-ray luminosity of high-mass X-ray binaries.

An alternative view: spherically symmetric accretion

Alternatively, accretion flow in a spherically symmetric case can be described by the same equations as that used for the coronal flow,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho v\right) = 0,$$

$$\rho v\frac{\mathrm{d}v}{\mathrm{d}r} = -a^2\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{\rho GM}{r^2}.$$

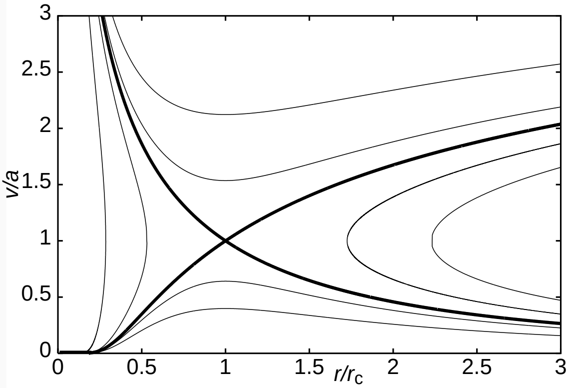
The integration of the continuity equation gives the **mass-accretion rate**

$$\dot{M} \equiv -4\pi r^2\rho v = \text{const.}$$

Inserting the continuity equation, the momentum equation takes the form

$$\frac{1}{v}\left(\frac{v^2}{a^2} - 1\right)\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2}{r} - \frac{GM}{r^2a^2}.$$

Solution of the Parker equation



There are two types of solutions describing outflow (wind) and inflow (accretion). There is one outflow solution that is subsonic at large distances from the star and supersonic close to the star corresponding to the accretion from stationary interstellar medium. Other inflow solution correspond to accretion of matter with non-zero velocity.

Analytical solution

The momentum equation

$$\frac{1}{v}\left(\frac{v^2}{a^2}-1\right)\frac{\mathrm{d}v}{\mathrm{d}r}=\frac{2}{r}-\frac{GM}{r^2a^2}$$

can be easily integrated. To do so, it is convenient to introduce a critical point radius

$$r_\mathrm{c}=\frac{GM}{2a^2}=\frac{1}{2}R_\mathrm{Bondi}(v=a),$$

using which the momentum equation reads (after integration)

$$\left(\frac{v}{a}\right)^2-\ln\left(\frac{v}{a}\right)^2=4\ln\frac{r}{r_\mathrm{c}}+4\frac{r_\mathrm{c}}{r}+K,$$

where K is an integration constant. Selecting the solution that smoothly passes throught the criticar point $v(r_\mathrm{c})=a$, $K=-3$ and

$$\left(\frac{v}{a}\right)^2-\ln\left(\frac{v}{a}\right)^2=4\ln\frac{r}{r_\mathrm{c}}+4\frac{r_\mathrm{c}}{r}-3.$$

Mass-accretion rate

For a subsonic flow $v \ll a$ at large distances from the star, $r \gg r_c$, the solution of the momentum equation can be simplified as

$$-\ln\left(\frac{v}{a}\right)^2 = 4\ln\frac{r}{r_c} - 3,$$

which gives for the velocity

$$v = -a\left(\frac{r_c}{r}\right)^2 e^{\frac{3}{2}}.$$

Here we selected a negative root, which gives positive
Bondi accretion rate

$$\dot{M} \equiv -4\pi r^2 \rho_\infty v = 4\pi e^{\frac{3}{2}} r_c^2 \rho_\infty a = \pi e^{\frac{3}{2}} \frac{(GM)^2}{a^3} \rho_\infty,$$

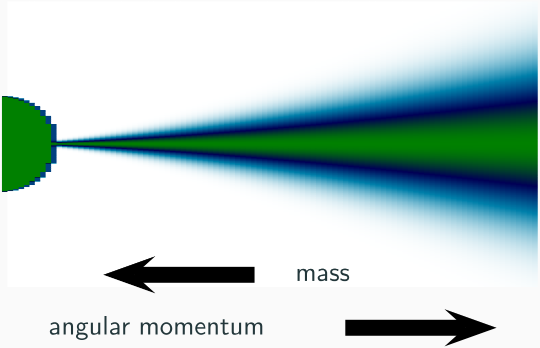
where ρ_∞ is the density of the medium at large distances from the star. The accretion rate is clearly proportional to $T^{-3/2}$.

Accretion disks in a nutshell

The need for angular momentum conservation

The very existence of accretion disks stems from the angular momentum conservation. Assuming that a typical blob of interstellar medium with mass $M = 1\,\mathrm{M}_\odot \approx 10^{33}\,\mathrm{g}$, radius $r_{\mathrm{ISM}} = 0.1\,\mathrm{pc} \approx 10^{17}\,\mathrm{cm}$, and velocity $v_{\mathrm{ISM}} = 1\,\mathrm{km\,s^{-1}} = 10^{5}\,\mathrm{cm\,s^{-1}}$ forms a star with about the same mass $M = 1\,\mathrm{M}_\odot \approx 10^{33}\,\mathrm{g}$ and radius $R_* = 1\,\mathrm{R}_\odot \approx 10^{11}\,\mathrm{cm}$, the need for the angular momentum conservation implies rotational velocities of the order of $r_{\mathrm{ISM}}\, v_{\mathrm{ISM}}/R_* = 10^{11}\,\mathrm{cm\,s^{-1}}$, which is significantly higher than the escape speed and in fact even higher than speed of the light. Therefore, during accretion, the matter has to get rid of angular momentum. As a result of universality of angular momentum conservation law, the accretion disks appear in different astrophysical environment including pre-main sequence stars, binaries with mass transfer, and centers of active galactic nuclei.

Mass and angular momentum transfer



Basic hydrodynamics of the disks

Assuming spherically symmetric outflow, the radial component of the momentum equation

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} + v_z \frac{\partial v_R}{\partial z} - \frac{v_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + g_R$$

can be in a stationary state ($\partial v_R / \partial t = 0$) neglecting the density gradient for a slow inflow ($v_R \ll v_\phi$) rewritten as

$$g_R + \frac{v_\phi^2}{R} = 0,$$

which in equatorial plane $g_R = -GM/R^2$ gives the Keplerian velocity

$$v_\phi = \sqrt{\frac{GM}{R}}.$$

Matter moves on Keplerian orbits in accretion disks. The angular velocity $\Omega = v_\phi / R \sim R^{-3/2}$ decreases with radius, whereas magnitude of the angular momentum per unit of mass $j = R v_\phi \sim R^{1/2}$ increases with radius. Consequently, matter losses angular momentum during accretion.

The need for anomalous viscosity

The momentum transfer in the direction perpendicular to the flow (orbital) motion requires viscosity. As a result of viscosity, the particles moving on neigbouring orbits may interchange the angular momentum. The coefficient of viscosity ν is given as a product of the particle mean free path ℓ and the mean velocity $\langle v \rangle$, $\nu \approx \ell \langle v \rangle$. The particle mean free path is given as $\ell = 1/(n\sigma)$, where n is particle number density and σ is the particle cross section.

The viscous time, a characteristic time of the angular momentum transfer, is given by $\tau \approx R^2/\nu$. For a characteristic radius $R = 1\,\mathrm{au} \approx 10^{13}\,\mathrm{cm}$ with a typical particle number density $n \approx 10^{14}\,\mathrm{cm}^{-3}$ and $\langle v \rangle \approx 10^5\,\mathrm{cm\,s}^{-1}$, for a molecular viscosity with $\sigma \approx 10^{-16}\,\mathrm{cm}^2$ the viscous time $\tau \approx 10^{19}\,\mathrm{s} \approx 10^{11}\,\mathrm{yr}$ is too large to generate a meaningfull accretion. Therefore, there is a need for additional *anomalous viscosity*.

Accretion disks in a more detail

Viscous disk equations: equation of continuity

Integrating the equation of continuity in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho v_R) + \frac{1}{R}\frac{\partial}{\partial \phi}(\rho v_\phi) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

over the vertical variable (z) assuming axisymmetric flow gives

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma v_R) = 0,$$

where

$$\Sigma = \int_{-\infty}^{\infty} \rho \, \mathrm{d}z$$

and where we assumed that v_R does not depend on z and that the disk disk density is zero in the limit $z \to \pm\infty$ (therefore $\int_{-\infty}^{\infty} \partial/\partial z \, (\rho v_z) = 0$).

Viscous disk equations: z component of the equation of motion

The z component of the equation of motion in cylindrical coordinates

$$\frac{\partial v_z}{\partial t} + v_R \frac{\partial v_z}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_z}{\partial \phi} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + g_z$$

gives with no vertical motion $v_z = 0$ equation of hydrostatic equilibrium

$$-\frac{\partial p}{\partial z} + \rho g_z = 0.$$

The z component of the gravity is using the first order Taylor expansion

$$g_z = -\frac{GM}{r^2} \frac{z}{r} = -\frac{GMz}{(R^2 + z^2)^{3/2}} \approx -\frac{GMz}{R^3}.$$

The equation of hydrostatic equilibrium can be integrated with $p = a^2 \rho$ and assuming (vertically) isothermal disk,

$$\rho(z) = \rho_0 e^{-\frac{GMz^2}{2a^2 R^3}} = \rho_0 e^{-\frac{1}{2} \frac{z^2}{H^2}},$$

which gives the Gaussian density distribution with scale-height $H = aR^{3/2}/(GM) = aR/v_\mathrm{K}$. The mid-plane disk density is related to the vertically integrated density via $\Sigma = \sqrt{2\pi} \rho_0 H$.

Viscous disk equations: R component of equation of motion

Integrating the radial component of the equation of motion in cylindrical coordinates

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} + v_z \frac{\partial v_R}{\partial z} - \frac{v_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + g_R$$

multiplied by ρ over the vertical variable (z) assuming axisymmetric flow gives

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} - \frac{v_\phi^2}{R} = -\frac{1}{\Sigma}\frac{\partial}{\partial R}\left(\int_{-\infty}^{\infty} p \, \mathrm{d}z\right) + \frac{1}{\Sigma}\int_{-\infty}^{\infty} \rho g_R \, \mathrm{d}z.$$

Introducing vertically independent isothermal speed of sound a in the first integral and assuming Gaussian vertical density distribution in the second integral with the Taylor expansion of g_R in terms of z

$$g_R = -\frac{GM}{r^2}\frac{R}{r} = -\frac{GMR}{(R^2+z^2)^{3/2}} \approx -\frac{GM}{R^2}\left(1 - \frac{3}{2}\frac{z^2}{R^2}\right)$$

gives after the integration over z the momentum equation

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} - \frac{v_\phi^2}{R} = -\frac{1}{\Sigma}\frac{\partial}{\partial R}(a^2\Sigma) - \frac{GM}{R^2} + \frac{3}{2}\frac{a^2}{R}.$$

Viscous disk equations: ϕ component of equation of motion

The Keplerian disk motion ($v_\phi \sim R^{-1/2}$) results in a radial shear. As a result of radial shear, fluid elements exchange angular momentum leading to a net flux of mass inwards and agular momentum outwards. Such angular momentum flux perpendicular to the mass flow is desribed by viscosity. However, the ϕ component of equation of motion in cylindrical coordinates,

$$\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} + v_z \frac{\partial v_\phi}{\partial z} + \frac{v_R v_\phi}{R} = -\frac{1}{\rho R}\frac{\partial p}{\partial \phi} + g_\phi,$$

which assuming axisymmetric flow and $g_\phi = 0$ reads as

$$\frac{\partial v_\phi}{\partial t} + \frac{v_R}{R}\frac{\partial (R v_\phi)}{\partial R} = 0,$$

was derived assuming inviscid flow and therefore does not contain any viscosity.

Introducing anomalous viscosity

The corresponding component of stress tensor

$$\tau_{R\phi} = \mu R \frac{\partial \Omega}{\partial R}$$

is parameterized assumimg proportionality to pressure via α parameter of viscosity (Shakura & Sunyaev 1973) as

$$\tau_{R\phi} = \alpha p.$$

The viscosity is assumed to stem mostly from the magnetorotational instability for which $\alpha \approx 0.01$.

The vertically integrated momentum equation is then

$$\frac{\partial v_\phi}{\partial t} + \frac{v_R}{R}\frac{\partial (R v_\phi)}{\partial R} + \frac{\alpha}{\Sigma R^2}\frac{\partial}{\partial R}\left(R^2 \int_{-\infty}^{\infty} p \, \mathrm{d}z\right) = 0,$$

or, with $\mu = \alpha a \rho H$ (assuming eddies with length-scale H and speed a)

$$\frac{\partial v_\phi}{\partial t} + \frac{v_R}{R}\frac{\partial (R v_\phi)}{\partial R} = \frac{\alpha}{\Sigma R^2}\frac{\partial}{\partial R}\left(a^2 \Sigma \frac{R^4}{v_\phi}\frac{\partial \left(\frac{v_\phi}{R}\right)}{\partial R}\right).$$

Apparently, angular momentum transfer is zero for rigidly rotating disks.

Stationary disks

In stationary ($\partial/\partial t = 0$) disk the continuity equation reads

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}(R\Sigma v_R) = 0,$$

which has a solution

$$2\pi R\Sigma v_R = \dot{M} = \text{const.}$$

giving the disk accretion rate.

The orbital velocity dominates the radial component of equation of motion $v_\phi \gg a \gg v_R$, therefore the equation again predicts the Keplerian velocity distribution close to the star

$$v_\phi = \sqrt{\frac{GM}{R}}.$$

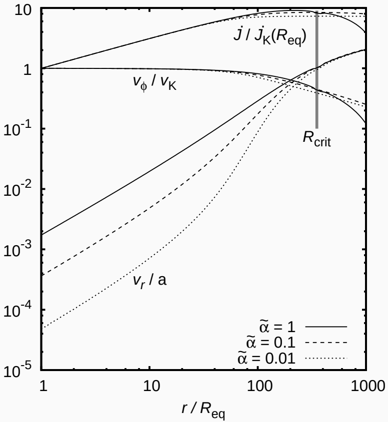
Stationary disks: example of solution

Equations for stationary disks

$$\frac{1}{R}\frac{\mathrm{d}(R\Sigma v_R)}{\mathrm{d}R} = 0,$$

$$v_R\frac{\mathrm{d}v_R}{\mathrm{d}R}+\frac{1}{\Sigma}\frac{\mathrm{d}(a^2\Sigma)}{\mathrm{d}R}+\frac{GM}{R^2} = \frac{v_\phi^2+\frac{3}{2}a^2}{R},$$

$$v_R\frac{\mathrm{d}(Rv_\phi)}{\mathrm{d}R} = \frac{\alpha}{\Sigma R}\frac{\mathrm{d}}{\mathrm{d}R}\left[a^2\Sigma\frac{R^4}{v_\phi}\frac{\mathrm{d}\left(\frac{v_\phi}{R}\right)}{\mathrm{d}R}\right]$$



predict nearly Keplerian rotation close to the star, decrease of the accretion velocity and of the angular momentum $\dot{J} = \dot{M}v_\phi R$ towards the star. The disks are subsonic ($v_R < a$) up to the critical radius R_{crit}.

Multiplication of the azimuthal component of the momentum equation by $2\pi R\,\Sigma$ gives for Keplerian rotation the accretion rate $\dot{M} = 3\pi\alpha a^2\Sigma R/v_\phi$.

Disk evolution

Assuming the Keplerian rotation, the ϕ-component of the momentum equation yields equation for disk radial velocity

$$v_R = -\frac{3\alpha}{\Sigma R v_\phi} \frac{\partial}{\partial R}(R^2 a^2 \Sigma).$$

Inserting this expression into the continuity equation gives

$$\frac{\partial \Sigma}{\partial t} = \frac{3\alpha}{R} \frac{\partial}{\partial R} \left(\frac{1}{v_\phi} \frac{\partial}{\partial R}(R^2 a^2 \Sigma) \right),$$

which is a diffusion equation for Σ. From this equation the viscous time-scale is

$$\frac{\partial \Sigma}{\partial t} \approx \frac{\Sigma}{\tau_{\text{vis}}} \approx \frac{\alpha a^2 \Sigma}{R v_\phi},$$

or

$$\tau_{\text{vis}} = \frac{R v_\phi}{\alpha a^2} = \frac{1}{\alpha \Omega} \left(\frac{R}{H} \right)^2.$$

The viscous time-scale is of the order of years for stellar disks ($\alpha \approx 1$). However, this equation is typically used to derive α from observations.

Thermodynamics of the disk

Disk structural equations

The vertical structure of the optically thick disk can be described by a similar structural equations as the equations used for the modelling of stellar structure or stellar atmospheres, i.e., hydrostatic equilibrium equation

$$\frac{\partial p}{\partial z} = -\rho \frac{GMz}{R^3},$$

heat transport equation

$$F = \begin{cases} -\frac{4acT^3}{3\kappa\rho}\frac{\partial T}{\partial z}, & \nabla \equiv \frac{\mathrm{d}\ln T}{\mathrm{d}\ln p} < \nabla_{\mathrm{add}}, \\ F_{\mathrm{conv}}, & \nabla > \nabla_{\mathrm{add}}, \end{cases}$$

and the equation describing the frictional generation of heat

$$\frac{\partial F}{\partial z} = \mu \left(R \frac{\mathrm{d}\Omega}{\mathrm{d}R} \right)^2.$$

These equations permit to get the thermal structure of the disk and to predict disk spectra solving the radiation transfer equation.

Some general relations

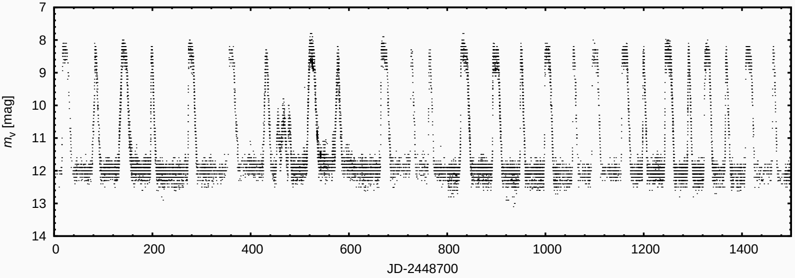
Integration of the equation for $\partial F / \partial z$ from the disk midplane upwards and assuming Keplerian disk rotation gives for the flux from the surface and effective disk temperature

$$F_0 = \sigma T_{\text{eff}}^4 = \frac{9}{8} \frac{GM}{R^2} \frac{\alpha a^2 \Sigma}{v_K}.$$

The accretion leads to heating, which can lead to the emission of high-energy photons from the disk (UV, X-ray).

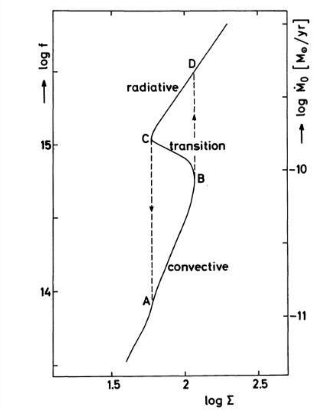
Application to cataclysmic variables

Cataclysmic variables are binary systems in which a white dwarf accretes matter through an accretionm disk form a late-type main sequence companion, which fills its Roche lobe. These systems show semi-regular outbursts in which the brightness rises by several magnitudes.



Light curve of SS Cyg derived from AAVSO database

Outburst in cataclysmic variables



Disk structural equations allow to obtain relation between the disk column density Σ and the accretion rate. In the region of partial ionization of hydrogen and helium this relation is double valued and allows two solutions connected by an unstable branch. This leads to cyclical behaviour (Meyer & Meyer-Hofmeister 1982).

Self-gravitating accretion disks

The disk gravity

In many cases, the mass of the disk is comparable to the mass of the central object (e.g., AGN disks and disks around young stellar objects). In such a case, the gravitational potential Φ in the equation of motion is given as a sum of two terms corresponding to the contribution of the central object and the disk, $\Phi = \Phi_c + \Phi_d$. For geometrically thin disk, the potential can be obtained from the Poission's equation in the form of

$$\nabla^2 \Phi_d = 4\pi G \Sigma \delta(z),$$

where $\delta(z)$ is the Dirac δ-function.

Disk orbital velocity

Again, assuming spherically symmetric inflow, the radial component of the momentum equation

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} + v_z \frac{\partial v_R}{\partial z} - \frac{v_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{\partial \Phi}{\partial R}$$

can be in a stationary state ($\partial v_R/\partial t = 0$) neglecting the density gradient for a slow inflow ($v_R \ll v_\phi$) rewritten as

$$\frac{v_\phi^2}{R} = \frac{\partial \Phi}{\partial R}.$$

For self-graviting disks with $\Sigma = \Sigma_0 R_0/R$ (Mestel 1963) the Poission's equations gives (Lodato 2007)

$$\frac{\partial \Phi_\mathrm{d}}{\partial R} = 2\pi G \Sigma.$$

In this case the orbital velocity is non-Keplerian,

$$v_\phi = \sqrt{1 + \frac{2\pi \Sigma_0 R_0}{M} R}\, v_\mathrm{K}.$$

This implies $v_\phi =$ const. for strongly self-graviting disks.

Disk vertical structure

For radially homogeneous slab we have the Poission's equation

$$\frac{\partial^2 \Phi_\mathrm{d}}{\partial z^2} = 4\pi G \rho,$$

which after integration from $-z$ to z gives

$$\frac{\partial \Phi_\mathrm{d}(z)}{\partial z} = 2\pi G \Sigma(z),$$

where $\Sigma(z) = \int_{-z}^{z} \rho(z')\,\mathrm{d}z'$. Therefore, the hydrostatic equilibrium equation in the vertical direction is

$$-\frac{a^2}{\rho}\frac{\partial \rho}{\partial z} = 2\pi G \Sigma(z).$$

This equation has a solution

$$\rho(z) = \frac{\rho_0}{\cosh^2(z/H_\mathrm{sg})}, \qquad H_\mathrm{sg} = \frac{a^2}{\pi G \Sigma}.$$

When the self-gravity becomes important?

The vertical component of the gravitational field produced by the disk is of the order of

$$2\pi G\Sigma \approx \frac{GM_{\mathrm{disk}}}{R^2},$$

while taking into account the projection effect the vertical component of the gravitational field of the central object is of the order of

$$\frac{GM}{R^2}\frac{H}{R}.$$

Therefore, the self-gravity of the disk becomes important already when

$$\frac{M_{\mathrm{disk}}}{M} \approx \frac{H}{R},$$

which is typically much smaller than 1. Therefore, the disk self-gravity may become important even if the disk mass is smaller than the mass of the central object.

Suggested reading

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