Magnetohydrodynamic waves

Again starting with simple

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Langmuir waves

Langmuir waves (plasma oscillations)

In ionized plasma, lightweight electrons move around heavy nuclei. There is a new type of oscillations connected with this motion. Within the limit of plasma oscillations (Langmuir waves) we can consider nuclei as static and assume that electrons oscilate in the field of nuclei.

These oscillations can be described by hydrodynamic equations describing free electrons

$$\frac{\partial \rho_{\mathrm{e}}}{\partial t} + \nabla \cdot (\rho_{\mathrm{e}} \boldsymbol{v}_{\mathrm{e}}) = 0,$$

$$\rho_{\mathrm{e}} \frac{\partial \boldsymbol{v}_{\mathrm{e}}}{\partial t} + \rho_{\mathrm{e}} \boldsymbol{v}_{\mathrm{e}} \cdot \nabla \boldsymbol{v}_{\mathrm{e}} = -e n_{\mathrm{e}} \boldsymbol{E},$$

where $\boldsymbol{E}$ is the electric field induced by free electrons and n_{e} is electron density.

Langmuir waves: perturbations

As always, we shall introduce perturbations of electron density and velocity in the comoving frame

$$\rho_{\mathrm{e}} = \rho_{0\mathrm{e}} + \delta\rho_{\mathrm{e}},$$

$$\boldsymbol{v}_{\mathrm{e}} = \delta\boldsymbol{v}_{\mathrm{e}},$$

for which the hydrodynamic equations are (neglecting higher order terms)

$$\frac{\partial\delta\rho_{\mathrm{e}}}{\partial t} + \rho_{0\mathrm{e}}\,\mathrm{div}\,\delta\boldsymbol{v}_{\mathrm{e}} = 0,$$

$$\frac{\partial\delta\boldsymbol{v}_{\mathrm{e}}}{\partial t} = -\frac{e}{m_{\mathrm{e}}}\boldsymbol{E}.$$

The electric field comes from the inhomogeneous distribution of density for which the Poisson equation gives

$$\mathrm{div}\,\boldsymbol{E} = \frac{4\pi e}{m_{\mathrm{e}}}\delta\rho_{\mathrm{e}}.$$

Langmuir waves: the wave equation

All these equations of perturbations shall be combined. Calculating divergence of momentum equation and taking $\operatorname{div}\delta\boldsymbol{v}$ from the continuity equation and $\boldsymbol{E}$ from the Poisson equation we derive the wave equation

$$\frac{\partial^2\rho_\mathrm{e}}{\partial t^2} + \frac{4\pi e^2}{m_\mathrm{e}^2}\rho_{0\mathrm{e}}\delta\rho_\mathrm{e} = 0,$$

which describes Langmuir waves. This gives the plasma frequency

$$\omega_\mathrm{pl}^2 = \frac{4\pi e^2}{m_\mathrm{e}} n_\mathrm{e}.$$

The waves with $\omega \lesssim \omega_\mathrm{pl}$ interact with free electrons and force the electrons to follow the electron motion. Therefore, the waves for frequencies $\omega \lesssim \omega_\mathrm{pl}$ are damped in plasma. On the contrary, the electrons are not able to react on higher frequency waves $\omega \gtrsim \omega_\mathrm{pl}$ and such waves move freely in plasma.

Alfvén waves

Alfvén waves: the ansatz

To start with truly MHD waves, we shall study perturbations in incompressible gas ($\operatorname{div} \boldsymbol{v} = 0$) in the comoving frame

$$\boldsymbol{v} = \delta \boldsymbol{v},$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \delta \boldsymbol{B}.$$

We shall start with momentum equation

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \boldsymbol{f} + \frac{1}{4\pi} \left(\operatorname{rot} \boldsymbol{B} \right) \times \boldsymbol{B},$$

which in absence of external forces and considering the above mentioned perturbations

$$\rho \frac{\partial \delta \boldsymbol{v}}{\partial t} = \frac{1}{4\pi} \left(\operatorname{rot} \delta \boldsymbol{B} \right) \times \boldsymbol{B}_0,$$

where we have assumed that derivatives of perturbations dominate.

Alfvén waves: perturbing the magnetic field

Similarly, the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{rot}\left(\boldsymbol{v} \times \boldsymbol{B}\right)$$

transforms into

$$\frac{\partial \delta \boldsymbol{B}}{\partial t} = \operatorname{rot}\left(\delta \boldsymbol{v} \times \boldsymbol{B}_0\right).$$

Using the corresponding vector identity

$$\operatorname{rot}\left(\delta \boldsymbol{v} \times \boldsymbol{B}_0\right) = \delta \boldsymbol{v} \operatorname{div} \boldsymbol{B}_0 - \boldsymbol{B}_0 \operatorname{div} \delta \boldsymbol{v} + (\boldsymbol{B}_0 \nabla)\delta \boldsymbol{v} - (\delta \boldsymbol{v} \nabla)\boldsymbol{B}_0.$$

Most terms on the right hand side of this equation can be dropped. The first one from the Maxwell equations, the second one from the incompressibility of the gas, and the last one due to the dominance of wave perturbvations. As a result,

$$\frac{\partial \delta \boldsymbol{B}}{\partial t} = (\boldsymbol{B}_0 \nabla)\delta \boldsymbol{v}.$$

Alfvén waves: the wave equation

Combining the resulting momentum and induction equations, we arrive at the wave equation for $\delta\boldsymbol{v}$

$$\rho\frac{\partial^2\delta\boldsymbol{v}}{\partial t^2} = \frac{1}{4\pi}\,\mathrm{rot}\,\left[(\boldsymbol{B}_0\nabla)\delta\boldsymbol{v}\right]\times\boldsymbol{B}_0.$$

We can assume that the magnetic field $\boldsymbol{B}_0$ is parallel with z-axis. In such case the wave equation takes the form

$$4\pi\rho\begin{pmatrix}\frac{\partial^2\delta v_x}{\partial t^2}\\ \frac{\partial^2\delta v_y}{\partial t^2}\\ \frac{\partial^2\delta v_z}{\partial t^2}\end{pmatrix} = B_0^2\begin{pmatrix}\frac{\partial^2\delta v_x}{\partial z^2}-\frac{\partial^2\delta v_z}{\partial x\partial z}\\ \frac{\partial^2\delta v_y}{\partial z^2}-\frac{\partial^2\delta v_z}{\partial y\partial z}\\ 0\end{pmatrix}.$$

The z-component gives $\partial^2\delta v_z/\partial t^2 = 0$, which means that there are no oscillations in the direction of magnetic field. With initial conditions $\delta v_z|_{t=0} = 0$ and $\partial\delta v_z/\partial t|_{t=0} = 0$ this gives $\delta v_z = 0$.

Alfvén velocity

With zero perturbation velocity in the z-axis direction, the wave equation simplifies to

$$\frac{B_0^2}{4\pi\rho}\begin{pmatrix} \frac{\partial^2 \delta v_x}{\partial z^2} \\ \frac{\partial^2 \delta v_y}{\partial z^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \delta v_x}{\partial t^2} \\ \frac{\partial^2 \delta v_y}{\partial t^2} \\ 0 \end{pmatrix}.$$

This corresponds to transversal waves (Alfvén waves) moving in the direction of magnetic field (z-axis) with so-called Alfvén velocity

$$v_\mathrm{A} = \frac{B_0}{\sqrt{4\pi\rho}}.$$

Alfvén waves: the induction equation

Similarly, from the time derivative of the induction equation

$$\frac{\partial^2 \delta \boldsymbol{B}}{\partial t^2} = (\boldsymbol{B}_0 \nabla) \frac{\partial \delta \boldsymbol{v}}{\partial t} = \frac{1}{4\pi\rho} (\boldsymbol{B}_0 \nabla) \left[(\mathrm{rot}\, \delta \boldsymbol{B}) \times \boldsymbol{B}_0 \right].$$

Again, assuming that the magnetic field $\boldsymbol{B}_0$ is parallel with z-axis, the wave equation takes the form of

$$4\pi\rho \begin{pmatrix} \frac{\partial^2 \delta B_x}{\partial t^2} \\ \frac{\partial^2 \delta B_y}{\partial t^2} \\ \frac{\partial^2 \delta B_z}{\partial t^2} \end{pmatrix} = B_0^2 \begin{pmatrix} \frac{\partial^2 \delta B_x}{\partial z^2} - \frac{\partial^2 \delta B_z}{\partial x \partial z} \\ \frac{\partial^2 \delta B_y}{\partial z^2} - \frac{\partial^2 \delta B_z}{\partial y \partial z} \\ 0 \end{pmatrix}.$$

The z-component gives $\partial^2 \delta B_z / \partial t^2 = 0$, which means that there are no oscillations in the direction of magnetic field. With initial conditions $\delta B_z|_{t=0} = 0$ and $\partial \delta B_z / \partial t|_{t=0} = 0$ this gives $\delta B_z = 0$.

Alfvén velocity

With zero magnetic field perturbation in the z-axis direction, the wave equation simplifies to

$$\frac{B_0^2}{4\pi\rho}\begin{pmatrix} \frac{\partial^2 \delta B_x}{\partial z^2} \\ \frac{\partial^2 \delta B_y}{\partial z^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \delta B_x}{\partial t^2} \\ \frac{\partial^2 \delta B_y}{\partial t^2} \\ 0 \end{pmatrix}.$$

This also corresponds to transversal Alfvén waves moving in the direction of magnetic field (z-axis) with Alfvén velocity

$$v_\mathrm{A} = \frac{B_0}{\sqrt{4\pi\rho}}.$$

Magnetoacoustic waves

Magnetoacoustic waves: the ansatz

We shall study a general form of MHD waves allowing for compression of gas. As always, we shall study perturbations in the comoving frame

$$\rho = \rho + \delta\rho,$$

$$\boldsymbol{v} = \delta\boldsymbol{v},$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}.$$

The MHD equations for perturbations are

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\boldsymbol{v} = 0,$$

$$\frac{\partial\delta\boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0}\nabla\delta p + \frac{1}{4\pi\rho_0}\left(\mathrm{rot}\,\delta\boldsymbol{B}\right)\times\boldsymbol{B}_0, \quad \delta p = a^2\delta\rho,$$

$$\frac{\partial\delta\boldsymbol{B}}{\partial t} = \mathrm{rot}\left(\delta\boldsymbol{v}\times\boldsymbol{B}_0\right).$$

Magnetoacoustic waves: velocity perturbations

Taking derivative of momentum equation with respect of time and insterting from the remaining equations

$$\frac{\partial^2 \delta \boldsymbol{v}}{\partial t^2} = a^2 \nabla \operatorname{div} \delta \boldsymbol{v} + \frac{1}{4\pi\rho_0} \operatorname{rot} \left[\operatorname{rot} \left(\delta \boldsymbol{v} \times \boldsymbol{B}_0 \right) \right] \times \boldsymbol{B}_0,$$

or, using the vector identity

$$\frac{\partial^2 \delta \boldsymbol{v}}{\partial t^2} = a^2 \nabla \operatorname{div} \delta \boldsymbol{v} + \frac{1}{4\pi\rho_0} \left[\nabla \operatorname{div} \left(\delta \boldsymbol{v} \times \boldsymbol{B}_0 \right) - \Delta \left(\delta \boldsymbol{v} \times \boldsymbol{B}_0 \right) \right] \times \boldsymbol{B}_0.$$

Because $\operatorname{div} \left(\delta \boldsymbol{v} \times \boldsymbol{B}_0 \right) = \boldsymbol{B}_0 \cdot \operatorname{rot} \delta \boldsymbol{v} - \delta \boldsymbol{v} \cdot \operatorname{rot} \boldsymbol{B}_0 \approx \boldsymbol{B}_0 \cdot \operatorname{rot} \delta \boldsymbol{v}$, we have $\nabla \left(\boldsymbol{B}_0 \cdot \operatorname{rot} \delta \boldsymbol{v} \right) =$ $\left(\boldsymbol{B}_0 \cdot \nabla \right) \operatorname{rot} \delta \boldsymbol{v} + \left(\operatorname{rot} \delta \boldsymbol{v} \cdot \nabla \right) \boldsymbol{B}_0 + \boldsymbol{B}_0 \times \operatorname{rot} \operatorname{rot} \delta \boldsymbol{v} + \operatorname{rot} \delta \boldsymbol{v} \times \operatorname{rot} \boldsymbol{B}_0 \approx$ $\left(\boldsymbol{B}_0 \cdot \nabla \right) \operatorname{rot} \delta \boldsymbol{v} + \boldsymbol{B}_0 \times \operatorname{rot} \operatorname{rot} \delta \boldsymbol{v} =$ $\left(\boldsymbol{B}_0 \cdot \nabla \right) \operatorname{rot} \delta \boldsymbol{v} + \boldsymbol{B}_0 \times \left(\nabla \operatorname{div} \delta \boldsymbol{v} - \Delta \delta \boldsymbol{v} \right)$. The term with Δ cancels out and $\left(\boldsymbol{B}_0 \times \nabla \operatorname{div} \delta \boldsymbol{v} \right) \times \boldsymbol{B}_0 = B_0^2 \, \nabla \operatorname{div} \delta \boldsymbol{v} - \left(\boldsymbol{B}_0 \cdot \nabla \operatorname{div} \delta \boldsymbol{v} \right) \cdot \boldsymbol{B}_0$.

Magnetoacoustic waves: the dispersion relation

Finally, we arrive at the wave equation in the form of

$$\frac{\partial^2 \delta \boldsymbol{v}}{\partial t^2} = \left[a^2 + \frac{B_0^2}{4\pi\rho_0}\right] \nabla \mathrm{div}\, \delta \boldsymbol{v} + \frac{1}{4\pi\rho_0} \left\{ \left[(\boldsymbol{B}_0 \cdot \nabla)\, \mathrm{rot}\, \delta \boldsymbol{v} \right] \times \boldsymbol{B}_0 - (\boldsymbol{B}_0 \cdot \nabla \mathrm{div}\, \delta \boldsymbol{v}) \cdot \boldsymbol{B}_0 \right\} .$$

We will seek the solution in the form of travelling waves

$$\delta \boldsymbol{v} \sim \exp\left[i(\boldsymbol{k}\boldsymbol{r} - \omega t) \right] .$$

In this case $\mathrm{div}\, \delta \boldsymbol{v} = i\boldsymbol{k} \cdot \delta \boldsymbol{v}$, $\nabla \mathrm{div}\, \delta \boldsymbol{v} = -(\boldsymbol{k} \cdot \delta \boldsymbol{v})\, \boldsymbol{k}$, $\mathrm{rot}\, \delta \boldsymbol{v} = i\boldsymbol{k} \times \delta \boldsymbol{v}$, $(\boldsymbol{B}_0 \cdot \nabla)\, \mathrm{rot}\, \delta \boldsymbol{v} = -(\boldsymbol{B}_0 \cdot \boldsymbol{k})\, (\boldsymbol{k} \times \delta \boldsymbol{v})$, and the dispersion relation is

$$\omega^2 \delta \boldsymbol{v} = (a^2 + v_{\mathrm{A}}^2)\, (\boldsymbol{k} \cdot \delta \boldsymbol{v})\, \boldsymbol{k} + \frac{1}{4\pi\rho_0} \left[(\boldsymbol{B}_0 \cdot \boldsymbol{k})\, (\boldsymbol{k} \times \delta \boldsymbol{v}) \times \boldsymbol{B}_0 - (\boldsymbol{B}_0 \cdot \boldsymbol{k})\, (\boldsymbol{k} \cdot \delta \boldsymbol{v})\, \boldsymbol{B}_0 \right] .$$

Inserting $\boldsymbol{B}_0 = B_0 \boldsymbol{b}$ with unity vector $\boldsymbol{b}$, and using the vector identity $(\boldsymbol{k} \times \delta \boldsymbol{v}) \times \boldsymbol{B}_0 = (\boldsymbol{B}_0 \cdot \boldsymbol{k})\, \delta \boldsymbol{v} - (\boldsymbol{B}_0 \cdot \delta \boldsymbol{v})\, \boldsymbol{k}$ we derive the dispersion relation

$$\omega^2 \delta \boldsymbol{v} = \left[(a^2 + v_{\mathrm{A}}^2)\, (\boldsymbol{k} \cdot \delta \boldsymbol{v}) - v_{\mathrm{A}}^2\, (\boldsymbol{k} \cdot \boldsymbol{b})\, (\delta \boldsymbol{v} \cdot \boldsymbol{b}) \right] \boldsymbol{k} + v_{\mathrm{A}}^2 \left[(\boldsymbol{k} \cdot \boldsymbol{b})^2 \delta \boldsymbol{v} - (\boldsymbol{k} \cdot \boldsymbol{b})\, (\delta \boldsymbol{v} \cdot \boldsymbol{k}) \boldsymbol{b} \right] .$$

Sanity check: the Alfvén waves

For incompressible flow $\boldsymbol{k} \cdot \delta \boldsymbol{v} = 0$ and the dispersion relation reads

$$\omega^2 \delta \boldsymbol{v} = v_{\mathrm{A}}^2 \left[(\boldsymbol{k} \cdot \boldsymbol{b})^2 \, \delta \boldsymbol{v} - (\boldsymbol{k} \cdot \boldsymbol{b}) (\delta \boldsymbol{v} \cdot \boldsymbol{b}) \, \boldsymbol{k} \right].$$

Multiplying the wave equation by $\boldsymbol{k}$ and using the incompressible flow condition we have $(\boldsymbol{k} \cdot \boldsymbol{b}) (\delta \boldsymbol{v} \cdot \boldsymbol{b}) = 0$. Consequently, either $\boldsymbol{k} \cdot \boldsymbol{b} = 0$ or $\delta \boldsymbol{v} \cdot \boldsymbol{b} = 0$. However, the application of the former in the wave equation would preclude any oscillations, therefore

$$\delta \boldsymbol{v} \cdot \boldsymbol{b} = 0,$$

that is, the oscillations are perpendicular to the magnetic fields, as we have already seen. Consequently, the dispersion relation for the Alfvén waves is

$$\omega = \pm v_{\mathrm{A}} (\boldsymbol{k} \cdot \boldsymbol{b}).$$

The group velocity is

$$\boldsymbol{v}_{\mathrm{g}} = \nabla_k \, \omega \equiv \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z} \right) = \pm v_{\mathrm{A}} \boldsymbol{b}.$$

Therefore, the waves move in the direction of the magnetic field.

General dispersion relation

Multyplying the dispersion relation by $\boldsymbol{k}$ the last two right hand terms calcel out, and

$$\omega^2 \left(\boldsymbol{k} \cdot \delta \boldsymbol{v}\right) = \left(a^2 + v_{\mathrm{A}}^2\right) \left(\boldsymbol{k} \cdot \delta \boldsymbol{v}\right) k^2 - v_{\mathrm{A}}^2 \left(\boldsymbol{k} \cdot \boldsymbol{b}\right) \left(\delta \boldsymbol{v} \cdot \boldsymbol{b}\right) k^2 .$$

The term $\delta \boldsymbol{v} \cdot \boldsymbol{b}$ can be derived by multiplication of the dispersion relation by $\boldsymbol{b}$, after which surprisingly nearly all term vanish and we simply get

$$\omega^2 \left(\boldsymbol{b} \cdot \delta \boldsymbol{v}\right) = a^2 \left(\boldsymbol{k} \cdot \delta \boldsymbol{v}\right) \left(\boldsymbol{k} \cdot \boldsymbol{b}\right) .$$

Inserting this into the former relation and calceling out $\boldsymbol{k} \cdot \delta \boldsymbol{v}$ we finally arrive at the dispersion relation

$$\omega^4 - \omega^2 (a^2 + v_{\mathrm{A}}^2) k^2 + v_{\mathrm{A}}^2 a^2 \left(\boldsymbol{k} \cdot \boldsymbol{b}\right)^2 k^2 = 0.$$

Denoting $\boldsymbol{k} \cdot \boldsymbol{b} = k \cos\theta$, where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{b}$, the wave speed is

$$v_{\mathrm{f,s}}^2 = \frac{\omega^2}{k^2} = \frac{1}{2} \left[a^2 + v_{\mathrm{A}}^2 \pm \sqrt{(a^2 + v_{\mathrm{A}}^2)^2 - 4 v_{\mathrm{A}}^2 a^2 \cos^2\theta} \right]$$

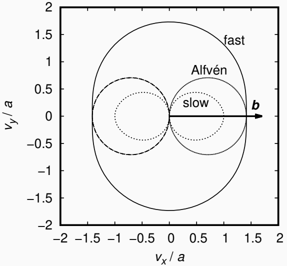
for fast and slow magnetoacoustic waves.

The dispersion relation of magnetoacoustic waves

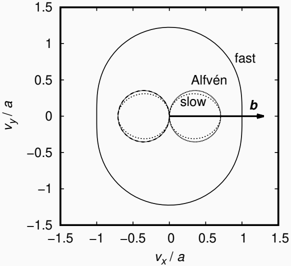
For $v_A \gg a$ the fast magnetoacoustic wave propagates nearly isotropically $v_f \approx v_A$, while the slow magnetoacoustic wave propagates at the velocity $v_s \approx a\cos\theta$. The energy propagates with group velocity $\boldsymbol{v}_g = \nabla_k \omega$, which is $v_g = v_A$ for the fast wave and $\boldsymbol{v}_g = a\boldsymbol{b}$ for the slow wave.

In a general case the dispersion relation depends just on a parameter $\beta = v_A^2/a^2$,

$$\frac{v_{f,s}^2}{a^2} = \frac{1}{2}\left[1+\beta \pm \sqrt{(1+\beta)^2 - 4\beta\cos^2\theta}\right].$$



$\beta = 2$



$\beta = 0.5$

Suggested reading

E. Battaner: Astrohysical fluid dynamics

L. Mestel: Stellar magnetism

L. C. Woods: Physics of Plasmas