Shock waves

The devil is in the detail

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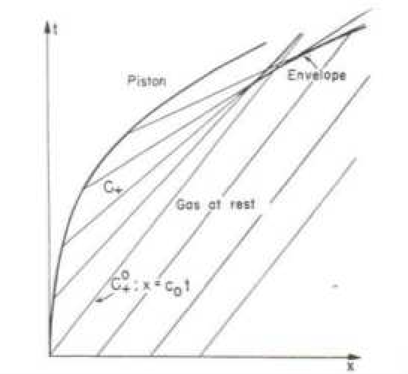
Motivation

Compression of a gas: advancing piston

Let us study a tube filled with a gas bounded by a piston advancing subsonically.

The piston starts at O and moves towards righ causing an compression of the gas. The gas adjanced to a piston moves with the same velocity as the piston. Only one set of characteristics drawn from piston propagates into the flow. The flow in a zone I is not influenced by a moving piston. Intersecting characteristic form an envelope. The solution is not unique at the intersection. This leads to a formation of a *shock wave*.



Shock conditions

Laws of conservation

Laws of conservation of mass, momentum, and energy do not necessary assume continuity of flow variables. A discontinuity can be regarded as the limiting case of very large gradients of the flow variables across a very thin layer. Viscosity an heat conduction may become important within such a thin layer. Nonviscid flow equations are not valid in this region.

Laws of conservation

Let us consider a column of gas bounded by $a_0(t) < x < a_1(t)$ containing a discontinuity. The conservation of mass requires

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0(t)}^{a_1(t)} \rho \,\mathrm{d}x = 0.$$

The change of momentum is given by forces acting on the column

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0(t)}^{a_1(t)} \rho v \,\mathrm{d}x = p(a_0, t) - p(a_1, t).$$

The change of energy is given by work of acting forces

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0(t)}^{a_1(t)} \rho \left(\frac{1}{2}u^2 + e\right) \mathrm{d}x = p(a_0, t)u(a_0, t) - p(a_1, t)u(a_1, t).$$

The entropy shall not decrease:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0(t)}^{a_1(t)} \rho s \geq 0.$$

Limit of thin layer

Assuming that the discontinuity apprears at coordinates $x = \xi(t)$ and moves with speed $\dot{\xi}(t) = U(t)$, the integrals of type

$$J = \int_{a_0(t)}^{a_1(t)} \psi(x,t)\,\mathrm{d}x$$

change with time as

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{a_0(t)}^{\xi(t)} \psi(x,t)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\xi(t)}^{a_1(t)} \psi(x,t)\,\mathrm{d}x$$

$$= \int_{a_0(t)}^{a_1(t)} \psi_t(x,t)\,\mathrm{d}x + \psi_0\dot{\xi}(t) - \psi(a_0,t)v(a_0,t) + \psi(a_1,t)v(a_1,t) - \psi_1\dot{\xi}(t).$$

Therefore, in the limit of $a_0(t) \to a_1(t)$, we have

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \psi_1 V_1 - \psi_0 V_0,$$

where $V_i = v_i - U$ are relative velocities with respect to shock.

Rankine-Hugoniot jump conditions

The application on the laws of conservation of mass, momentum, and energy gives the Rankine-Hugoniot jump conditions

$$\rho_1 V_1 - \rho_0 V_0 = 0,$$

$$\rho_1 v_1 V_1 - \rho_0 v_0 V_0 = p_0 - p_1,$$
$$\rho_1 \left(\frac{1}{2} v_1^2 + e_1\right) V_1 - \rho_0 \left(\frac{1}{2} v_0^2 + e_0\right) V_0 = p_0 v_0 - p_1 v_1,$$
$$\rho_1 s_1 V_1 - \rho_0 s_0 V_0 \geq 0.$$

These equation can cast in the conservative form

$$\rho_1 V_1 = \rho_0 V_0,$$

$$\rho_1 V_1^2 + p_1 = \rho_0 V_0^2 + p_0,$$
$$\frac{1}{2} V_1^2 + h_1 = \frac{1}{2} V_0^2 + h_0,$$
$$s_1 \geq s_0,$$

where we used equation of continuity, selected the coordinate frame, where the shock is stationary, $U = 0$, and where $h = e + p/\rho$.

Rankine-Hugoniot jump conditions for perfect gas

For a perfect gas with constant γ the specific enthalpy is $h = \gamma/(\gamma - 1)p/\rho$ and Rankine-Hugoniot jump conditions give

$$\frac{\rho_1}{\rho_0} = \frac{(\gamma + 1)M_0^2}{\gamma + 1 + (\gamma - 1)(M_0^2 - 1)} = \frac{V_0}{V_1},$$

$$\frac{p_1}{p_0} = \frac{\gamma + 1 + 2\gamma(M_0^2 - 1)}{\gamma + 1},$$

$$\frac{T_1}{T_0} = \frac{\left[\gamma + 1 + 2\gamma(M_0^2 - 1)\right]\left[\gamma + 1 + (\gamma - 1)(M_0^2 - 1)\right]}{(\gamma + 1)^2 M_0^2},$$

where the Mach number is $M_0 = V_0/a_0$ and the speed of sound is $a_0^2 = \gamma p_0/\rho_0$. These relations express the post-shock quantities ρ_1, V_1, p_1, and T_1 in terms of pre-schock quantities and M_0.

Compression shocks

Compression shocks

For compression shocks $M_0 > 1$ from the Rankine-Hugoniot jump conditions follows that $p_1 > p_0,\ T_1 > T_0$ (examine T_1/T_0 as a function of p_1/p_0, where we have substituted $M_0^2 = (\gamma+1)/(2\gamma)(p_1/p_0 - 1) + 1$, this shows that T_1/T_0 is monotonically increasing function with $T_1/T_0 = 1$ for $p_1/p_0 = 1$), $\rho_1 > \rho_0$ (ρ_1 has minimum as a function of M_0 for $M_0 > 1$ at $M_0 = 1$) and therefore $V_1 < V_0$. Therefore also

$$\frac{M_1^2}{M_0^2} = \frac{V_1^2}{V_0^2}\frac{a_0^2}{a_1^2} = \frac{V_1^2}{V_0^2}\frac{p_0}{p_1}\frac{\rho_1}{\rho_0} = \frac{V_1^2}{V_0^2}\frac{T_0}{T_1} < 1,$$

and $M_1 < 1$. Gas flows into the discontinuity with a supersonic velocity $M_0 > 1$ and flows out with a subsonic velocity $M_1 < 1$. Equivalently, the shock propagates at a supersonic velocity with respect to the undisturbed gas and at a subsonic velocity with respect to compressed gas behind the shock.

Infinitely strong shock $M_0 \to \infty$

In the limit of infinitely strong shocks ($M_0 \to \infty$) the Rankine-Hugoniot jump conditions give $p_1 \to \infty$ and $T_1 \to \infty$ (extremely high pressure and temperature), but the density remains finite

$$\lim_{M_0 \to \infty} \frac{\rho_1}{\rho_0} = \frac{\gamma + 1}{\gamma - 1},$$

which gives $\rho_1/\rho_0 = 4$ pro monoatomic perfect gas with $\gamma = 5/3$.

For very strong shocks $M_0 \gg 1$, the post-shock temperature is

$$T_1 = \frac{2\gamma(\gamma - 1)M_0^2}{(\gamma + 1)^2} T_0.$$

Since $M_0 = V_0/a_0$ and $a_0^2 = \gamma k T/\mu$, the post-shock temperature does not depend on the pre-shock temperature. Evaluating the post-shock temperature for a monoatomic gas with $\gamma = 5/3$,

$$T_1 = \frac{3}{16} \frac{\mu V_0^2}{k} = 14\,\mathrm{MK} \left(\frac{V_0}{1000\,\mathrm{km\,s^{-1}}} \right)^2.$$

Rarefaction shocks

Rarefaction shocks

Another type of shocks allowed by Rankine-Hugoniot jump conditions are rarefaction shocks, in which the existence of discontinuities leads to the expansion of the gas rather than to the compression. For $M_0 < 1$ we have $p_1 < p_0$, $T_1 < T_0$, $\rho_1 < \rho_0$ and therefore $V_1 > V_0$ and using the same reasoning $M_1 > 1$. Gas flows into the discontinuity with a subsonic velocity $M_0 < 1$ and flows out with a supersonic velocity $M_1 > 1$.

The change of entropy

The entropy of the gas with constant specific heats is $S = c_V \ln(p\rho^{-\gamma})$, therefore the entropy jump is

$$S_1 - S_0 = c_v \ln \frac{p_1 \rho_0^\gamma}{p_2 \rho_1^\gamma}$$

From the Rankine-Hugoniot jump conditions we have

$$M_0^2 = \frac{\gamma + 1}{2\gamma} \left(\frac{p_1}{p_0} - 1 \right) + 1$$

and inserting into the expression for $\rho_1/\rho 2$ follows that

$$S_1 - S_0 = c_v \ln \left\{ \frac{p_1}{p_0} \left[\frac{(\gamma - 1)p_1/p_0 + \gamma + 1}{(\gamma + 1)p_1/p_0 + \gamma - 1} \right]^\gamma \right\}.$$

The change of entropy

The analysis of behaviour of function $S_1 - S_0$ shows that the function is monotonic and that

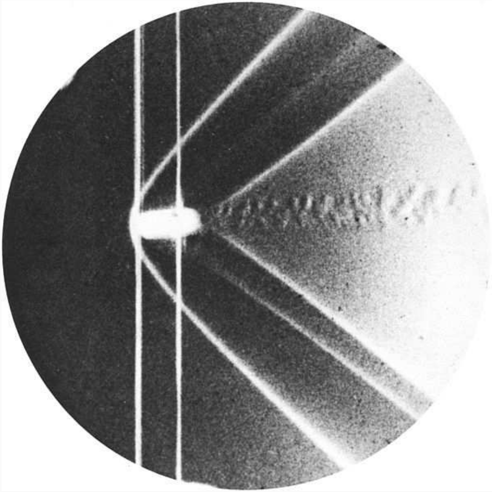
$$S_1 - S_0 \begin{cases} < 0, & \text{for } p_1 < p_0, \\ = 0, & \text{for } p_1 = p_0, \\ > 0, & \text{for } p_1 > p_0. \end{cases}$$

The entropy increases across the compression shock, whereas the entropy decreases across the rarefaction shock. Therefore, the rarefaction shocks are not termodynamically allowed. This statement holds for majority of real fluids.

Rarefaction shocks are not mechanically stable: any distaurbance in the pre-shocked gas travels with speed of sound and outruns the shock wave. After a certain time the rarefaction region will include the gas in front of the discontinuity and the discontinuity will dissapear.

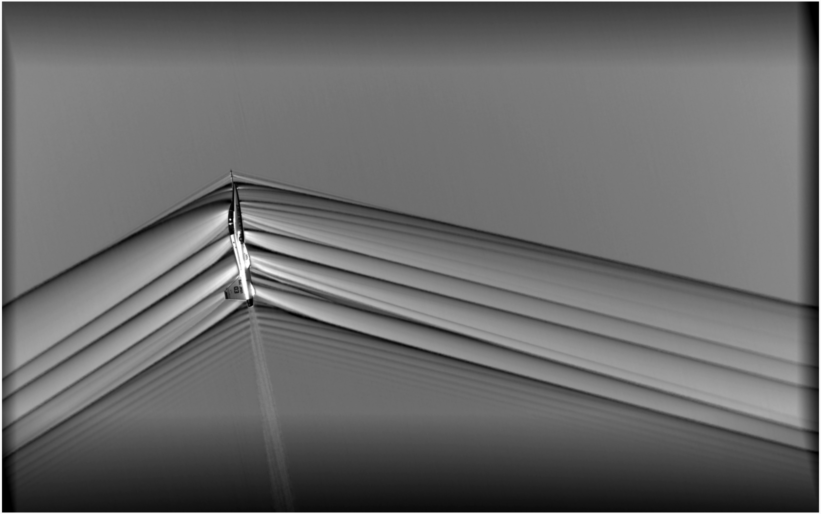
Shocks are everywhere...

Mach cone around bullet



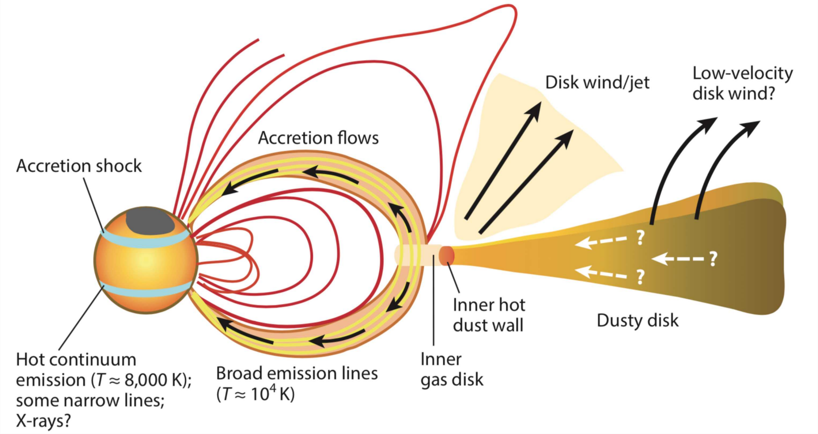
Schlieren photography of bullet taken in Prague by Mach (1887).

Mach cone around supersonic planes



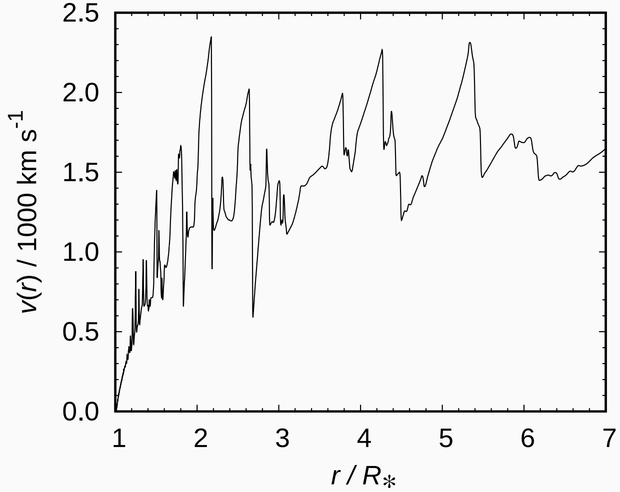
Schlieren photography of Mach cone around supersonic plane.

Initial accretion phase



(Hartmann 2016)

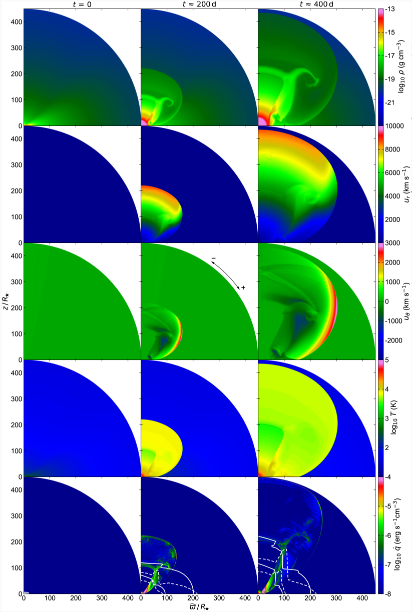
Shocks in hot star winds



(Feldmeier et al. 1997)

Interaction of stellar wind with interstellar environent

Interaction of supernovae with circumstellar environment



The plots show interaction of supernova blast wave with circumstellar environment in the form of disk (Kurfürst et al. 2020). The plots show density, radial and tangential velocity, and temperature in three time steps.

Suggested reading

R. Courant & K. O. Friedrichs: Supersonic Flow and Shock Waves

A. Feldmeier: Theoretical Fluid Dynamics

D. Mihalas & B. W. Mihalas: Foundations of Radiation Hydrodynamics

F. H. Shu: The physics of astrophysics: II. Hydrodynamics

Y. B. Zeldovich, Y. P. Raizer: Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena