Waves and instabilities

Starting from the simplest

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Sound waves

Sound waves

Let us assume that the hydrodynamical equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0,$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \rho \boldsymbol{g},$$

have static solution $\rho_0 = \text{const.}$ with $\boldsymbol{g} = 0$.

Sound waves

Let us search for a small perturbation $\delta\rho \ll \rho_0$ of the static solution in a form of $\rho = \rho_0 + \delta\rho$ and $\boldsymbol{v} = \delta\boldsymbol{v}$, which fullfills the hydrodynamical equations:

$$\frac{\partial(\rho_0 + \delta\rho)}{\partial t} + \nabla\cdot((\rho_0 + \delta\rho)\delta\boldsymbol{v}) = 0,$$

$$(\rho_0 + \delta\rho)\frac{\partial\delta\boldsymbol{v}}{\partial t} + (\rho_0 + \delta\rho)\delta\boldsymbol{v}\cdot\nabla\delta\boldsymbol{v} = -\nabla(p_0 + \delta p) + (\rho_0 + \delta\rho)\boldsymbol{g}.$$

Neglecting second-order terms we derive

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\boldsymbol{v} = 0,$$

$$\rho_0\frac{\partial\delta\boldsymbol{v}}{\partial t} + \nabla\delta p = 0.$$

Derivating the first equation with respect to t, inserting from the second one and rewritting $\delta p = \frac{\mathrm{d}p}{\mathrm{d}\rho}\delta\rho \equiv a^2\delta\rho$ we arrive at the wave equation

$$\frac{\partial^2\delta\rho}{\partial t^2} - a^2\nabla^2\delta\rho = 0.$$

The sound speed

The constant in the wave equation is the *sound speed*:

$$a = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}}.$$

For isothermal perturbations we derive from the perfect gas equation of state

$$a = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}} = \sqrt{\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_T} = \sqrt{\frac{kT}{\mu m_{\mathrm{H}}}},$$

where μ is the mean molecular weight and m_{H} is the mass of hydrogen atom. For fully ionized hydrogen $\mu = \frac{1}{2}$ and $a = \sqrt{2kT/(m_{\mathrm{H}})}$.

For adiabatic perturbations we have

$$a = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}} = \sqrt{\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_S} = \sqrt{\frac{\kappa kT}{\mu m_{\mathrm{H}}}},$$

where κ is the specific heat ratio. For fully ionized hydrogen $a = \sqrt{10kT/(5m_{\mathrm{H}})}$.

Characteristics of differential equations

Characteristic direction

Let us assume that $f = f(x, y)$. Then a linear combination $af_x + bf_y$ (where $f_x = \partial f/\partial x$) is a *directional derivative* of f along the direction $\mathrm{d}x : \mathrm{d}y = a : b$. If $(x(\sigma), y(\sigma))$ is a curve parameterized by σ, $x_\sigma : y_\sigma = a : b$, then $af_x + bf_y$ is a directional derivative along the curve.

Let us consider system of 2 equations for two functions $u(x, y)$, $v(x, y)$:

$$L_1 \equiv A_{11}u_x + B_{11}u_y + A_{12}v_x + B_{12}v_y + C_1 = 0,$$

$$L_2 \equiv A_{21}u_x + B_{21}u_y + A_{22}v_x + B_{22}v_y + C_2 = 0.$$

We ask for a linear combination

$$L = \lambda_1 L_1 + \lambda_2 L_2$$

so that in the differential expression L the derivatives of u and v combine to derivatives in the same direction. Such direction is characteristic.

Characteristic relations

Suppose that the characteristic direction is given by the above ratio $x_\sigma : y_\sigma$. Then the condition that u and v are differentiated in L in the same direction is

$$\lambda_1 A_{11}+\lambda_2 A_{21} : \lambda_1 B_{11}+\lambda_2 B_{21} = \lambda_1 A_{12}+\lambda_2 A_{22} : \lambda_1 B_{12}+\lambda_2 B_{22} = x_\sigma : y_\sigma .$$

This gives the system of equations for λ_1 and λ_2

$$\hat{M}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = 0, \text{ where } \hat{M} = \begin{pmatrix} A_{11}y_\sigma - B_{11}x_\sigma & A_{21}y_\sigma - B_{21}x_\sigma \\ A_{12}y_\sigma - B_{12}x_\sigma & A_{22}y_\sigma - B_{22}x_\sigma \end{pmatrix}$$

leading to *characteristic relations*. The system has a non-trivial solution if $\det \hat{M} = 0$. This gives equation in a form of

$$a y_\sigma^2 - 2b x_\sigma y_\sigma + c x_\sigma^2 = 0.$$

For $ac - b^2 > 0$, this cannot be satisfied by any direction. Such equations are called *elliptic*. For $ac - b^2 < 0$ we have two characteristic directions. Such systems are called *hyperbolic*. There are two sets of equations $\frac{\mathrm{d}y}{\mathrm{d}x} = \xi_+$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \xi_-$ defining two sets of characteristics C_+ and C_-.

Characteristic relations for 1D flow

For 1D flow $\rho = \rho(x,t) \equiv u$ and $v = v(x,t)$ the corresponding system is

$$L_1 \equiv \rho_t + v\rho_x + \rho v_x = 0,$$

$$L_2 \equiv v_t + v\,v_x + \frac{a^2}{\rho}\rho_x = 0.$$

This gives ($t \equiv y$)

$$\hat{M} = \begin{pmatrix} vt_\sigma - x_\sigma & \frac{a^2}{\rho}t_\sigma \\ \rho t_\sigma & vt_\sigma - x_\sigma \end{pmatrix}.$$

From $\det \hat{M} = 0$ the characteristic relation is $(vt_\sigma - x_\sigma)^2 - a^2 t_\sigma^2 = 0$, or

$$(v \pm a)t_\sigma = x_\sigma.$$

The characteristics correspond to sound waves.

Characteristic relations for 1D flow

The relation between λ_1 and λ_2 can be derived, e.g., from the first equation of the system $\hat{M}\boldsymbol{\lambda} = 0$

$$(vt_\sigma - x_\sigma)\lambda_1 + \frac{a^2}{\rho} t_\sigma \lambda_2 = 0,$$

which, after insterting the characteristic relation, simplifies to

$$\lambda_1 = \pm\frac{a}{\rho}\lambda_2.$$

Therefore, the linear combination of hydrodynamical equations is

$$L = \lambda_1 L_1 + \lambda_2 L_2 = \frac{a}{\rho}\left[\pm\rho_t + (a \pm v)\rho_x\right] + v_t + (v \pm a)v_x = 0,$$

where we further selected $\lambda_2 = 1$. As we can see, ρ and v are differentiated in the same direction.

Transformation to ordinary differential equation

Because $\rho_\sigma = \rho_t t_\sigma + \rho_x x_\sigma = [\rho_t + (v \pm a)\rho_x]\, t_\sigma$ from the characteristic relation, by doing the linear combination we have transformed the original system of partial differential equations to a system of ordinary differential equation with new variables $\alpha \equiv \sigma$ (for $+$ root) and $\beta \equiv \sigma$ (for $-$ root)

$$\frac{a}{\rho}\rho_\alpha + v_\alpha = 0,$$

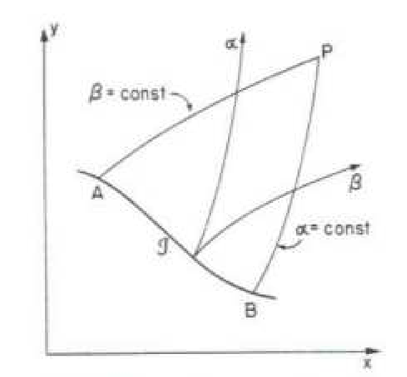
$$-\frac{a}{\rho}\rho_\beta + v_\beta = 0,$$

Domain of dependence

The solution of hydrodynamical equations define two sets of characteristics

$$(v \pm a)t_\sigma = x_\sigma .$$

Let us assume that the initial conditions are given on curve $\mathcal{J}$. There are two characteristics that go throught a selected point P. The line AB intercepted by the two characteristics is called *domain of dependence* of P. This can be utilized for a numerical integration.

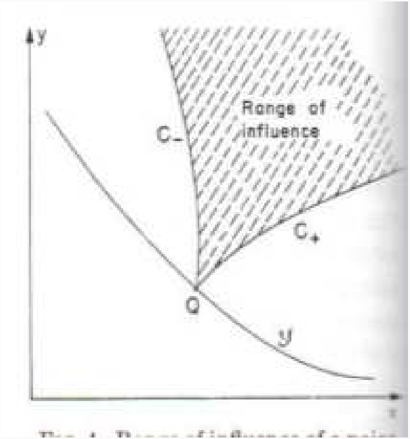


Range of influence

The solution of hydrodynamical equations define two sets of characteristics

$$(v \pm a)t_\sigma = x_\sigma.$$

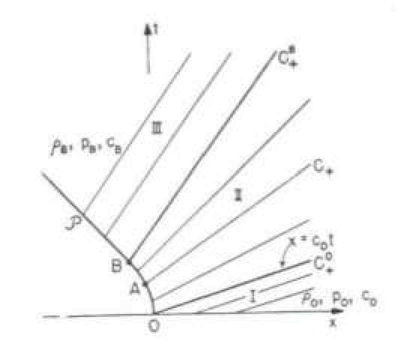
Let us assume that the initial conditions are given on curve $\mathcal{J}$. *The range of influence* of a point Q is the totality of points which are influenced by the initial data at the point Q. The range of influence of the point Q consists of all points P whose domain of dependence contains Q. The range of influence of influence is defined by two characteristic drawn through Q.



Expansion of a gas: withdrawing piston

Let us study a tube filled with a gas bounded by a piston withdrawing subsonically.

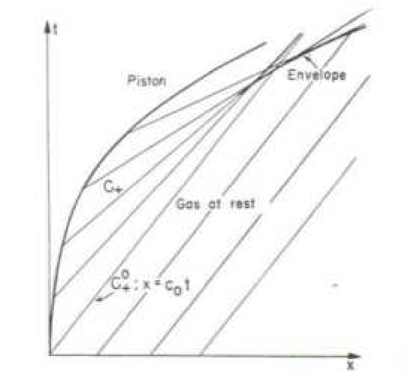
The piston starts at O and recedes towards left causing an expansion of the gas. The gas adjanced to a piston moves with the same velocity as the piston. Only one set of characteristics drawn from piston propagates into the flow. The flow in a zone I is not influenced by a moving piston. The flow in a zone II is a rarefaction wave.



Compression of a gas: advancing piston

Let us study a tube filled with a gas bounded by a piston advancing subsonically.

The piston starts at O and moves towards righ causing an compression of the gas. The gas adjanced to a piston moves with the same velocity as the piston. Only one set of characteristics drawn from piston propagates into the flow. The flow in a zone I is not influenced by a moving piston. Intersecting characteristic form an envelope. The solution is not unique at the intersection. This leads to a formation of a *shock wave*.



Characteristics of more than two equations

We consider n differential equations

$$L_i \equiv A_{ij}\frac{\partial u^j}{\partial x} + B_{ij}\frac{\partial u^j}{\partial y} + C_j, \qquad i = 1, \cdots, n.$$

We ask for a linear combination

$$L = \lambda_i L_i$$

so that in the differential expression L the derivatives of u^j combine to derivatives in the same direction. This gives the conditions

$$\lambda_i A_{ij} : \lambda_i B_{ij} = x_\sigma : y_\sigma, \qquad j = 1, \cdots, n,$$

or

$$\lambda_i (A_{ij} y_\sigma - B_{ij} x_\sigma) = 0, \qquad j = 1, \cdots, n.$$

This system has a non-trivial solution if

$$\det |A_{ij} y_\sigma - B_{ij} x_\sigma| = 0.$$

Application: Charactersistics including the energy equation

For isentropic 1D flow $\rho = \rho(x,t) \equiv u^1$, $v = v(x,t) \equiv u^2$, and $s = s(x,t) \equiv u^3$ the corresponding system of equations is

$$L_1 \equiv \rho_t + v\rho_x + \rho v_x = 0,$$

$$L_2 \equiv v_t + v\, v_x + \frac{a^2}{\rho}\rho_x = 0,$$

$$L_3 \equiv s_t + v s_x = 0.$$

$$\hat{M} = \begin{pmatrix} vt_\sigma - x_\sigma & \frac{a^2}{\rho} t_\sigma & 0 \\ \rho t_\sigma & vt_\sigma - x_\sigma & 0 \\ 0 & 0 & vt_\sigma - x_\sigma \end{pmatrix}.$$

From $\det \hat{M} = 0$ the first two characteristic relations are the same as without energy equation

$$(v \pm a)t_\sigma = x_\sigma,$$

$$vt_\sigma = x_\sigma.$$

This corresponds to sound waves propagating at the sound speed and entropy wave propagating at zero speed (with respect to the flow).

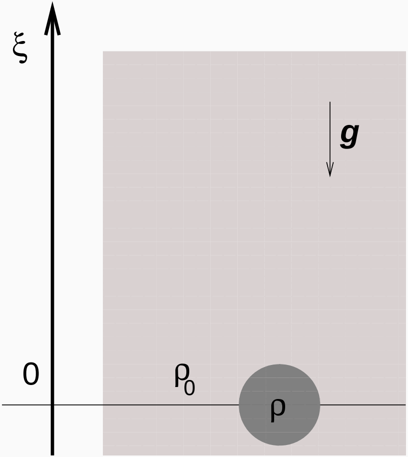
Gravity waves

Gravity waves

We will study the propagation of waves in an atmosphere, which is in hydrostatic equilibrium given by external gravitational field.

We will study the movement of a blob with density ρ in hydrostatic equilibrium with outside medium with density $\rho_0(\xi)$. We will assume the density gradient in the atmosphere $\left(\frac{d\rho}{dz}\right)_{at}$ and neglect the heat exchange between the blob and the atmosphere: the processes are adiabatic. The equation of motion including the buoyancy is:

$$\rho \frac{d^2\xi}{dt^2} = -g(\rho - \rho_0).$$



Gravity waves

In the equation of motion,

$$\rho \frac{\mathrm{d}^2 \xi}{\mathrm{d}t^2} = -g(\rho - \rho_0),$$

we shall use the Taylor expansion to derive the buoyancy term,

$$\rho_0(\xi) = \rho_0(0) + \left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{at}} \xi,$$

$$\rho(\xi) = \rho(0) + \left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{ad}} \xi.$$

Because the blob is initially in equilibrium, $\rho_0(0) = \rho(0)$, the equation of motion

$$\frac{\mathrm{d}^2 \xi}{\mathrm{d}t^2} = -\omega_{\mathrm{BV}}^2 \xi$$

describes an oscillatory motion, so-called gravity waves. The frequency of oscillations,

$$\omega_{\mathrm{BV}}^2 = \frac{g}{\rho} \left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{ad}} - \left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{at}} \right]$$

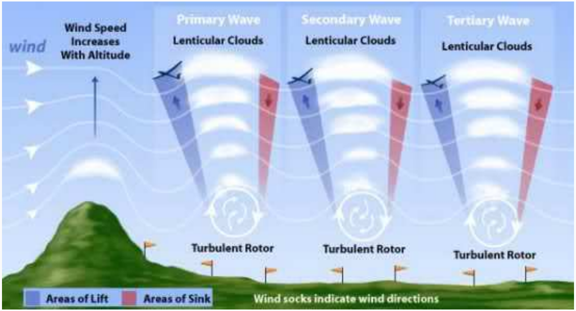
is the *Brunt-Väisälä frequency*.

Gravity waves: the case of $\omega_{\mathsf{BV}}^2 > 0$

For $\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{ad}} > \left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{at}}$, i.e., for $\left|\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{ad}}\right| < \left|\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{at}}\right|$ we have $\omega_{\mathrm{BV}}^2 > 0$.

The initial perturbation results in oscillations $\xi(t) = \xi_0 e^{\pm i|\omega_{\mathrm{BV}}|t}$.

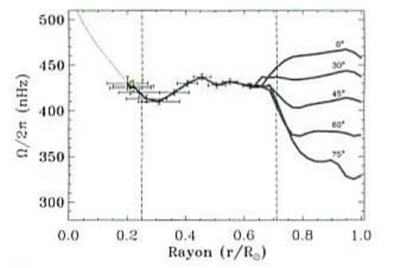
Gravity waves in Earth's atmosphere:



Gravity waves: the case of $\omega^2_{\mathsf{BV}} > 0$: Earth's atmosphere

Gravity waves: the case of $\omega_{\mathrm{BV}}^2 > 0$: Earth's atmosphere

Gravity waves: solar differential rotation



Angular velocity as a function of radius in the Sun from accoustic modes in heliseismic observations (Turck-Chiéze). The lack of differential rotation in the radiative zone is due to angular momentum transport by gravity waves (Charbonnel & Talon 2005).

Gravity waves: the case of $\omega_{\mathsf{BV}}^2 < 0$

For $\left(\frac{d\rho}{dz}\right)_{ad} < \left(\frac{d\rho}{dz}\right)_{at}$, i.e., for $\left|\left(\frac{d\rho}{dz}\right)_{ad}\right| > \left|\left(\frac{d\rho}{dz}\right)_{at}\right|$ we have $\omega_{BV}^2 < 0$.

The initial perturbation results in instability $\xi(t) = \xi_0 e^{\pm|\omega_{BV}|t}$.

The instability leads to convection.

Schwarzschild stability criterion

The stability criterion can be recast in another intuitive form. From the ideal gas equation of state,

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{at}} = \frac{\rho}{p}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)_{\mathrm{at}} - \frac{\rho}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_{\mathrm{at}} .$$

The convective plumes are in hydrostatic equilibrium with the sorrounding environment meaning that

$$\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)_{\mathrm{at}} \equiv \left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)_{\mathrm{ad}} = \gamma\frac{p}{\rho}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{ad}} .$$

Therefore, the stability criterion is

$$(1-\gamma)\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\mathrm{ad}} > -\frac{\rho}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_{\mathrm{at}} .$$

From the adiabatic equation follows that $(\mathrm{d}\rho/\mathrm{d}z)_{\mathrm{ad}} =$
$= 1/\left(\gamma-1\right)\rho/T\left(\mathrm{d}T/\mathrm{d}z\right)_{\mathrm{ad}}$, which yields

$$\left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_{\mathrm{ad}} < \left(\frac{\mathrm{d}T}{\mathrm{d}z}\right)_{\mathrm{at}} ,$$

which is Schwarzschild stability criterion.

Temperature distribution in a convective atmosphere

The convective motions are typically slower than the sound waves maintaining the hydrostatic equilibrium, therefore one can use

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g$$

to determine the temperature gradient. The pressure is $p = a^2\rho = kT\rho/\mu$, which gives

$$\frac{\mathrm{d}T}{\mathrm{d}r} + \frac{T}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{\mu g}{k}.$$

The adiabatic equation $T\rho^{1-\gamma} = \text{const.}$ gives

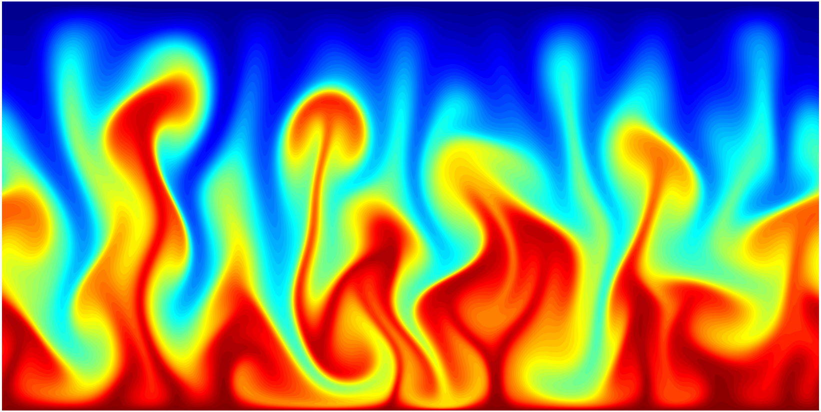
$$\frac{T}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{1}{\gamma - 1}\frac{\mathrm{d}T}{\mathrm{d}r},$$

which yields for the temperature gradient

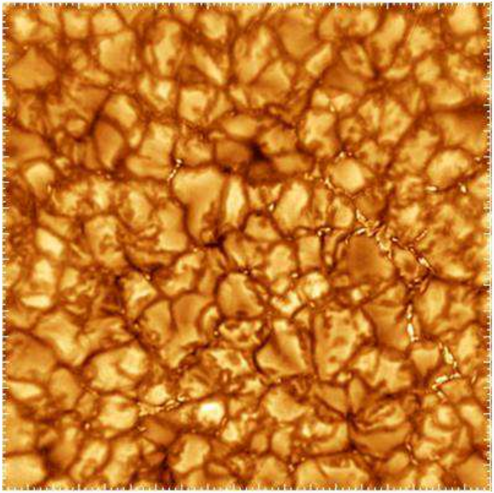
$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{\gamma - 1}{\gamma}\frac{\mu g}{k}.$$

This predicts the temperature gradient of $-10\,\mathrm{K\,km^{-1}}$ for the atmosphere of our Earth and about $5\times 10^6\,\mathrm{K}\,R_\odot^{-1}$ for the envelope of Sun.

Simulation of convection



Solar granulation



Kelvin-Helmholtz & Rayleigh-Taylor

K-H & R-T instabilities: the initial setup

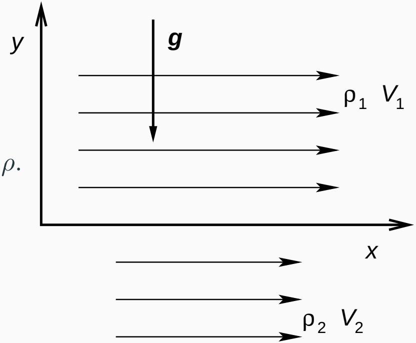
Consider a shear flow with velocity V_1 and density ρ_1 in the upper half plane and V_2 and ρ_2 in the lower half plane. We expect instability would occur within crossing time scale of the flow over the characteristic length scale. The surplus kinetic and potential energies proportional to $(V_2 - V_1)^2$ and $\rho_1 - \rho_2$ are the energy sources of turbulence.

The hydrodynamical equations are

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{v}) = 0,$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \boldsymbol{\nabla}\, \boldsymbol{v} - \rho \boldsymbol{g} = -\nabla p = -a^2 \nabla \rho.$$

We will assume an initial state with $\boldsymbol{v} = (V(y), 0, 0)$ and $\rho = \rho_0(y)$.



K-H & R-T instabilities: perturbing the initial state

We will assume the perturbed quantities in the form of $\boldsymbol{v} = (V(y) + \delta\tilde{v}_x, \delta\tilde{v}_y, 0)$ and $\rho = \rho_0(y) + \delta\tilde{\rho}$, where $\delta\tilde{v}_{x,\,y} \ll V(y)$ and $\delta\tilde{\rho} \ll \rho_0$. The perturbations are assumed to obey harmonical expansion

$$\delta\tilde{v}_{x,y} = \delta v_{x,y}(y)\exp(ikx + i\omega t),$$

$$\delta\tilde{\rho} = \delta\rho(y)\exp(ikx + i\omega t).$$

Therefore, we shall substitute $\partial/\partial t \to i\omega$ and $\boldsymbol{\nabla} \to (ik, \partial/\partial y, 0)$.

The linearized hydrodynamical equations are

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\boldsymbol{v}) = 0 \quad \to \quad \omega\delta\rho + Vk\delta\rho + \rho_0 k\delta v_x - i\rho_0\delta v_y' = 0,$$

$$\rho\frac{\partial v_x}{\partial t} + \rho v_i\frac{\partial v_x}{\partial x_i} = -a^2\frac{\partial\rho}{\partial x} \quad \to \quad \rho_0\omega\delta v_x + \rho_0 Vk\delta v_x - i\rho_0\delta v_y V' = -a^2 k\delta\rho,$$

$$\rho\frac{\partial v_y}{\partial t} + \rho v_i\frac{\partial v_y}{\partial x_i} = -a^2\frac{\partial\rho}{\partial y} - \rho g \quad \to \quad \rho_0\omega\delta v_y + \rho_0 Vk\delta v_y = ia^2\delta\rho' + i\delta\rho g,$$

where prime ($'$) denotes $\mathrm{d}/\mathrm{d}y$.

K-H & R-T instabilities: solving for perturbations

Solving the second equation for δv_x, inserting to the first one and solving for $\delta\rho$, and inserting the final relation to the last equation we derive (denoting $\omega_{\mathrm{d}} = \omega + kV$)

$$\rho_0\omega_{\mathrm{d}}\delta v_y = \left[a^2\frac{-\rho_0\delta v_y'\omega_{\mathrm{d}} + \rho_0 kV'\delta v_y}{\omega_{\mathrm{d}}^2 - k^2a^2}\right]' + g\frac{-\rho_0\delta v_y'\omega_{\mathrm{d}} + \rho_0 kV'\delta v_y}{\omega_{\mathrm{d}}^2 - k^2a^2}.$$

For relatively slow flow ($V \ll a$) we can assume $a \to \infty$ (incompresible flow) and the dispersion relation becomes

$$\left(\rho_0\omega_{\mathrm{d}}\delta v_y' - \rho_0 kV'\delta v_y\right)' - \rho_0\omega_{\mathrm{d}}k^2\delta v_y = 0.$$

K-H & R-T instabilities: back to the original problem

For an assumed velocity and density profile profile

$$V(y) = \begin{cases} V_1, & \text{for } y > 0, \\ V_2, & \text{for } y < 0, \end{cases} \qquad \rho_0(y) = \begin{cases} \rho_1, & \text{for } y > 0, \\ \rho_2, & \text{for } y < 0, \end{cases}$$

in each half-space. We have the dispersion relation $\delta v_y'' - k^2 \delta v_y = 0$ that has the solution (assuming $\delta v_y \to 0$ for $y \to \infty$)

$$\delta v_y \sim \begin{cases} \exp(-ky), & \text{for } y > 0, \\ \exp(ky), & \text{for } y < 0. \end{cases}$$

The displacement δy at the boundary should be continuous, consequently

$$\frac{\mathrm{D}\delta y}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)\delta y = \delta v_y$$

and therefore $v_y/(\omega + kV)$ shoud be continuous. The solution becomes

$$\delta v_y \sim \begin{cases} (\omega + kV_1)\exp(-ky), & \text{for } y > 0, \\ (\omega + kV_2)\exp(ky), & \text{for } y < 0. \end{cases}$$

Kelvin-Helmholtz instability

We assume constant density, no gravitational field, and velocity shear

$$V(y) = \begin{cases} V_1, & \text{for } y > 0, \\ V_2, & \text{for } y < 0. \end{cases}$$

From the requirement that the left-hand side of the dispersion relation

$$\left(\rho_0 \omega_\mathrm{d} \delta v_y' - \rho_0 k V' \delta v_y\right)' = \rho_0 \omega_\mathrm{d} k^2 \delta v_y$$

should be continuous at the boundary we have

$$\left(\rho_0 \omega_\mathrm{d} \delta v_y'\right)_1 = \left(\rho_0 \omega_\mathrm{d} \delta v_y'\right)_2,$$

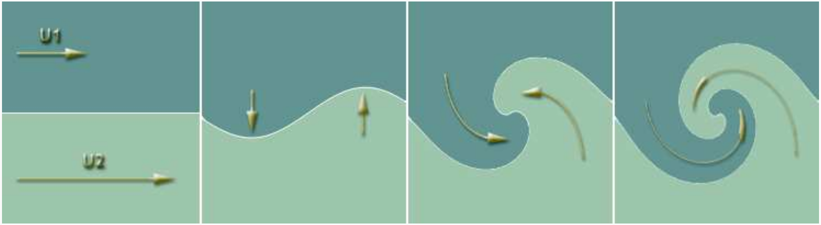
which, after substitution of ω_d and $\delta v_y'$ gives the dispersion relation

$$(\omega + kV_1)^2 + (\omega + kV_2)^2 = 0.$$

Solving for ω gives instability for $V_1 \neq V_2$:

$$\omega = -\frac{1}{2} k (V_1 + V_2) \pm \frac{1}{2} ik (V_1 - V_2).$$

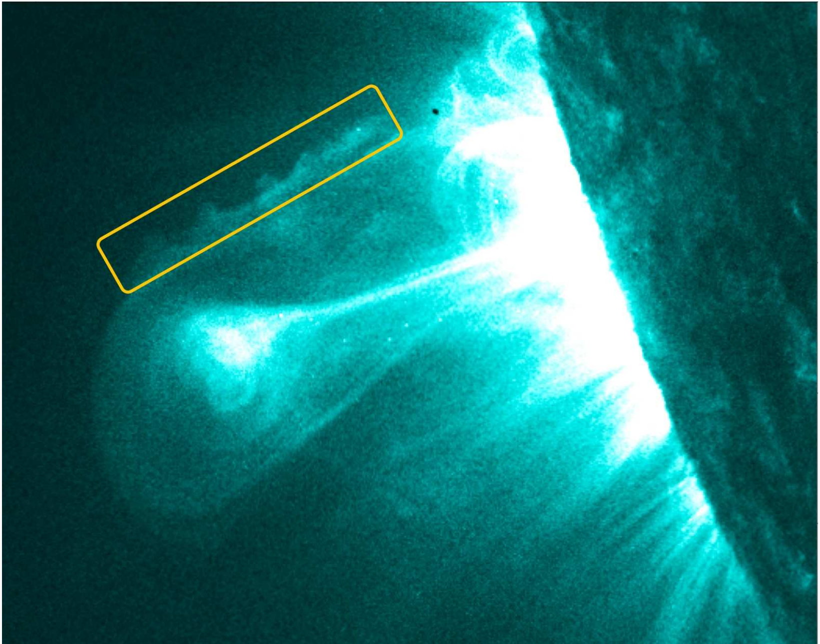
Kelvin-Helmholtz instability: going nonlinear



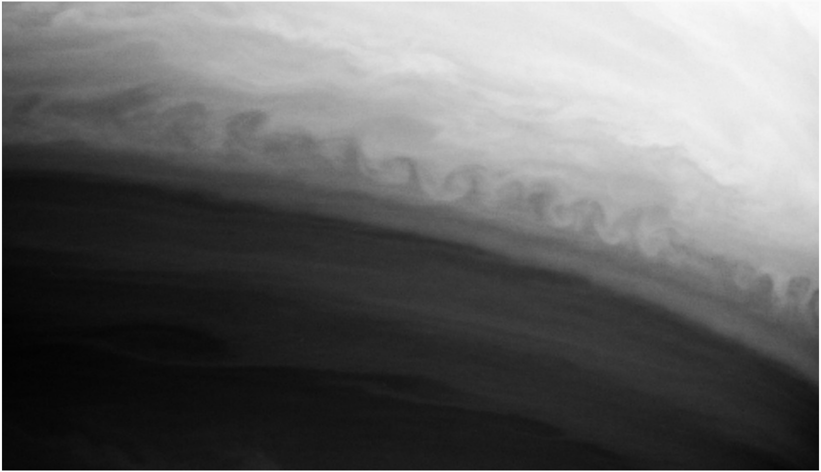
Kelvin-Helmholtz instability: Earth's atmosphere

Kelvin-Helmholtz instability: Earth's atmosphere

Kelvin-Helmholtz instability: Solar prominence



Kelvin-Helmholtz instability: Atmosphere of Saturn



Rayleigh-Taylor instability

We assume zero velocity $V(y)$ and the density

$$\rho_0(y) = \begin{cases} \rho_1, & \text{for } y > 0, \\ \rho_2, & \text{for } y < 0. \end{cases}$$

From the requirement that the left-hand side of the dispersion relation (assuming $\omega \gg k^2 a^2$)

$$\rho_0 \omega^2 \delta v_y + g \rho_0 \delta v_y' = \left[a^2 \frac{-\rho_0 \delta v_y'}{\omega_\mathrm{d}} \right]'$$

should be continuous at the boundary we have

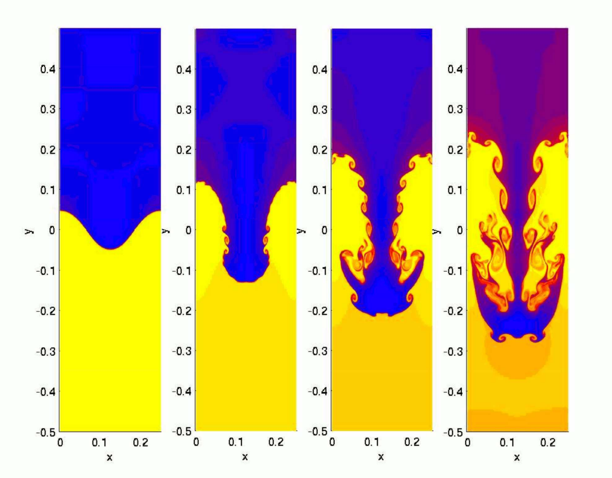
$$\left(\rho_0 \omega^2 \delta v_y + g \rho_0 \delta v_y'\right)_1 = \left(\rho_0 \omega^2 \delta v_y + g \rho_0 \delta v_y'\right)_2 .$$

Inserting the solution $\delta v_y \sim \exp(\pm k y)$ gives the dispersion relation

$$\omega^2 = g k \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}.$$

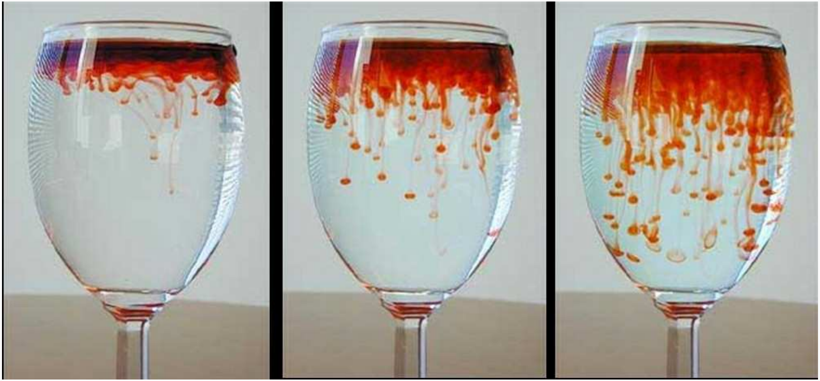
The flow is stable ($\omega^2 > 0$) for $\rho_2 > \rho_1$, while for $\rho_2 < \rho_1$ the Rayleigh-Taylor instability appears.

Fingers of Rayleigh-Taylor instability

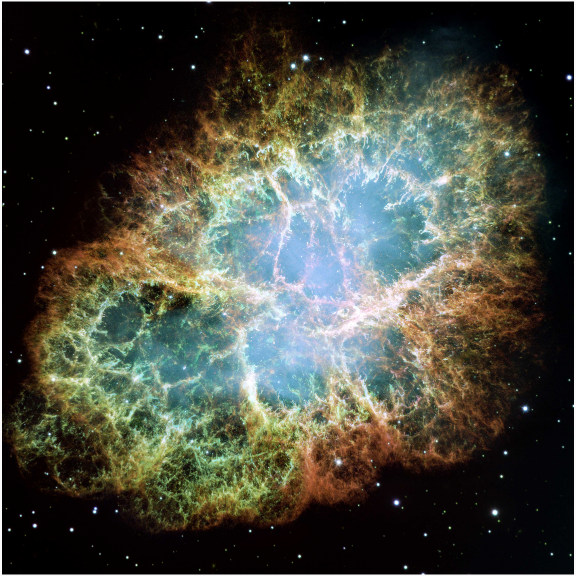


Figures show the developement of the finger typical for Rayleigh-Taylor instability. The instability is stabilized by the surface tension for large wavenumbers (Chandrasekhar). Figure shows also Kelvin-Helmholtz instabilities on the boundary of finger.

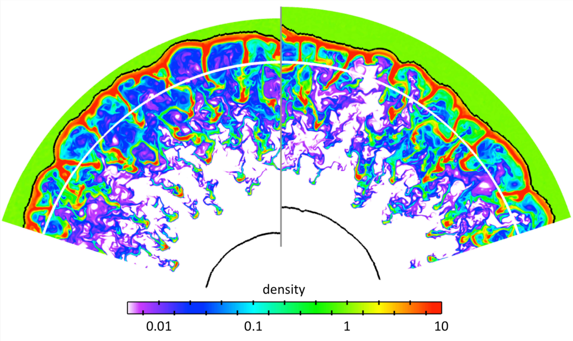
Visualisation of the Rayleigh-Taylor instability



Crab nebula



Crab nebula structure due to Rayleigh-Taylor instability



While the supernova nebula becomes flat, the swept-up, accelerating shell is subject to the RayleighTaylor instability (Kulsrud et al. 1965, Chevalier & Gull 1975, Blondin & Chevalier 2017).

Suggested reading

S. Chandrasekhar: Hydrodynamic and Hydromagnetic Stability

R. Courant & K. O. Friedrichs: Supersonic Flow and Shock Waves

A. Maeder: Physics, Formation and Evolution of Rotating Stars

D. Mihalas & B. W. Mihalas: Foundations of Radiation Hydrodynamics

F. H. Shu: The physics of astrophysics: II. Hydrodynamics

T. Tajima & K. Shibata: Plasma astrophysics

L. C. Woods: Physics of Plasmas