Cviceni 7: baze a dimenze

Monday, November 16, 2020 9:54 AM

Def: V - lin. prostor; $U, W \subset V$ - podpr;

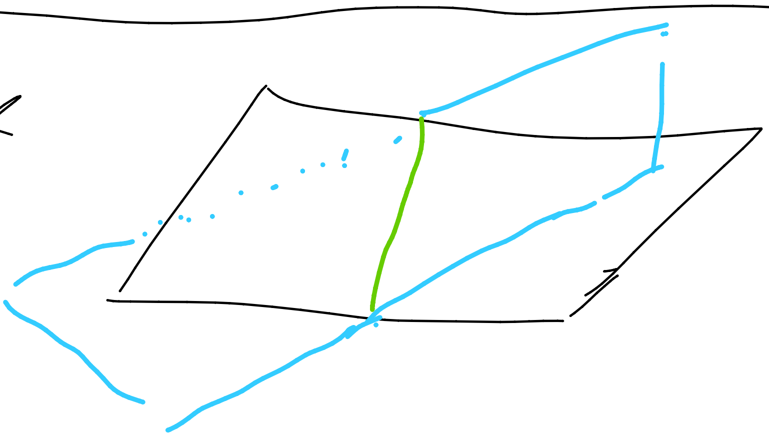
Součet: $U + W = \{z \in V : z = x + y, x \in U, y \in W\}$

Průnik: $U \cap W = \{z \in V : z \in U, z \in W\}$

$U + W$, $U \cap W$ - **podprostory** v V!

/ - průnik

Součet $= R^3$



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1) $V = R^3$, $U = \{x - y = 0\}$ $W = \{x - 2y = 0\}$

$U \cap W$: $\begin{cases} x - y = 0 \\ x - 2y = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 0, \\ y = 0 \\ z = t \end{cases}$ (osa z)

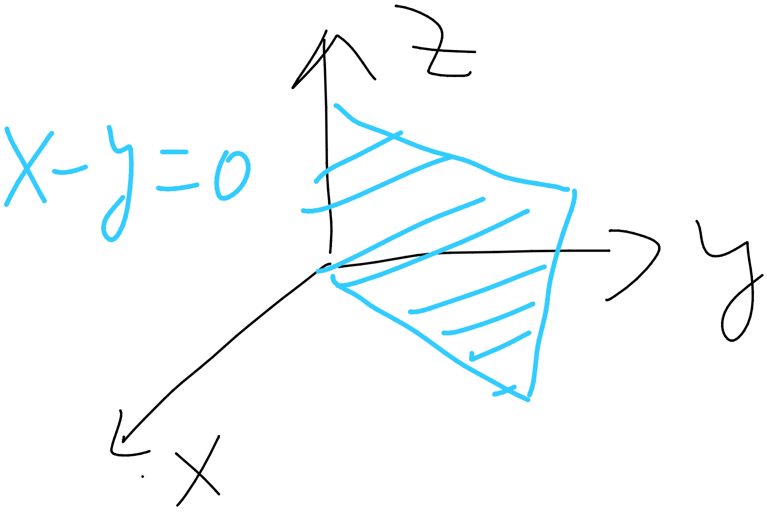
$U + W = R^3$.

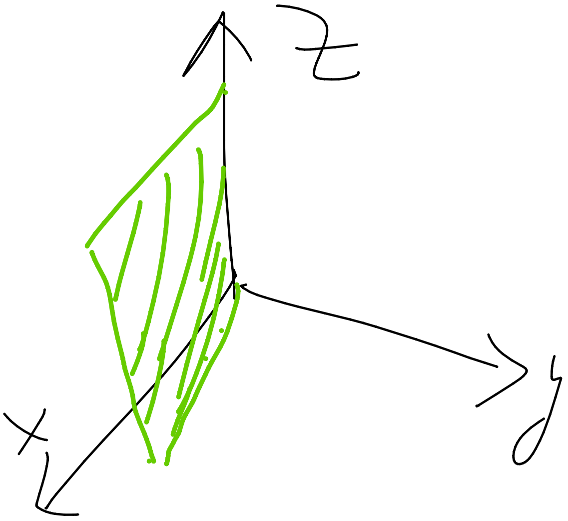
$U+W=\mathbb{R}^3$: $e_3=(0,0,1)\in U$

$$e_1=(1,0,0)=\underset{\overset{\Uparrow}{W}}{(2,1,0)}-\underset{\overset{\Uparrow}{U}}{(1,1,0)}$$

$$e_2=(0,1,0)=2\cdot\underset{\overset{\Uparrow}{U}}{(1,1,0)}-\underset{\overset{\Uparrow}{W}}{(2,1,0)}$$

$$\Rightarrow e_1,e_2,e_3\in U+W\Rightarrow U+W=\mathbb{R}^3$$





2) $V=\mathbb{R}_3[x]$

$U=\{p(x)=p(2x)\}$

$$W = \{ p(2x) = 2p(x) \}$$

$$U: a_0 + a_1 x + a_2 x^2 + a_3 x^3 =$$

$$= a_0 + 2a_1 x + 4a_2 x^2 + 8a_3 x^3$$

$$\Leftrightarrow a_1 = a_2 = a_3 = 0 \Rightarrow$$

$$U = [1]$$

$$W: a_0 + 2a_1 x + 4a_2 x^2 + 8a_3 x^3 =$$

$$= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3$$

$$\Leftrightarrow a_0 = a_2 = a_3 = 0 \Rightarrow$$

$$W = [x] \Rightarrow$$

$$U \cap W = \{0\}$$

$$U + W = [1, x] = \{a_0 + a_1 x\} = \mathbb{R}_1[x]$$

Vě: $\boxed{\dim U + \dim W = \dim(U+W) + \dim(U \cap W)}$

3) $V = \mathbb{R}^4$; $U = \left[\underset{\overset{\parallel}{a_1}}{\begin{pmatrix} 0 \\ -2 \\ -1 \\ -5 \end{pmatrix}}, \underset{\overset{\parallel}{a_2}}{\begin{pmatrix} -4 \\ -7 \\ -1 \\ -22 \end{pmatrix}} \right]$

$W = \left[\underset{\overset{\parallel}{a_3}}{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix}}, \underset{\overset{\parallel}{a_4}}{\begin{pmatrix} 7 \\ 10 \\ -5 \\ 43 \end{pmatrix}} \right]$

Najít baze a dim. pro $U+W$ a $U \cap W$.

$U+W = [U, W] = [a_1, a_2, a_3, a_4]$

Zapišme $x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4$

$$\begin{pmatrix} 0 & -4 & 1 & 7 \\ -2 & -7 & 2 & 10 \\ -1 & -1 & 1 & -5 \\ -5 & -22 & 5 & 43 \end{pmatrix} \xrightarrow{(1)} \begin{pmatrix} 1 & 1 & -1 & 5 \\ -2 & -7 & 2 & 10 \\ 0 & -4 & 1 & 7 \\ -5 & -22 & 5 & 43 \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & -5 & 0 & 20 \\ 0 & -4 & 1 & 7 \\ 0 & -17 & 0 & 68 \end{pmatrix} \xrightarrow{(3)}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & -4 & 1 & 7 \\ 0 & 1 & 0 & -4 \end{pmatrix} \xrightarrow{(4)} \begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -9 \end{pmatrix}$$

, x_4 - volna $\Rightarrow$

$a_4 \in [a_1, a_2, a_3]$, a taky

a_1, a_2, a_3 - lin. nezav.

$\Rightarrow U+W = [a_1, a_2, a_3]$

a_1, a_2, a_3 - baze v $U+W$

$\dim(U+W) = 3$

<u>Pro $U \cap W$</u>: $\dim U = 2$ (a_1, a_2 lin. nezav.)

$\dim W = 2 \Rightarrow \dim(U \cap W) = 1$

$U \cap W$: $x_1 a_1 + x_2 a_2 = x_3 a_3 + x_4 a_4$

$\iff x_1 a_1 + x_2 a_2 + (-x_3) a_3 + (-x_4) a_4 = 0$

$\Rightarrow$ používame <u>stejný</u> schodtvar!

Potřebujeme ∀ řeš (dim $U \cap W = 1$)

$\Rightarrow$ nechť $x_4 = -1 \Rightarrow x_1 = 0, x_2 = 4$

$\Rightarrow$ vypočítame $0 \cdot a_1 + 4 \cdot a_2 =$

$= 4a_2 \Rightarrow U \cap W = [a_2]$

a_2 - baze pro $U \cap W$

4) $V = \mathbb{R}^4$, U: $x_1 + x_2 + x_3 - x_4 = 0$

W: $x_1 - x_2 + x_3 + x_4 = 0$

Najít dim a baze $U + W$, $U \cap W$

Najít souřad. $a = (-1,1,1,1)$

B určenou Baze $U+W$, $U \cap W$.

<u>$U \cap W$</u>: $\begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 + x_4 = 0 \end{cases}$

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & -2 & 0 & 2 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$ x_3, x_4 - volny

$x_3 = 1, x_4 = 0 \Rightarrow (1, 0, -1, 0) = a_1$

$x_3 = 0, x_4 = 1 \Rightarrow (0, 1, 0, 1) = a_2$

$\forall X = s \cdot a_1 + t a_2 \Rightarrow \{a_1, a_2\}$ - Baze $U \cap W$

$\dim U \cap W = 2$, pak,

$\dim U = \dim W = 3$

(např. U: $(1,1,1,-1) \Rightarrow 3$ volných

$\Rightarrow \dim = 3$)

$\Rightarrow \dim = 3)$

$\Rightarrow \dim(U+W) = 3+3-2 = 4 \Rightarrow$

$U+W = R^4 = V$, Baze např. vybereme $\underline{\text{stand.}}$ e_1, e_2, e_3, e_4

$a = (-1, 1, 1, 1)$ в Baze $U+W$:

$\underline{(-1, 1, 1, 1)}$

a в Baze $U \cap W$:

$a = x_1 \cdot (1, 0, -1, 0) + x_2 \cdot (0, 1, 0, 1)$

$\Rightarrow a = -x_1 + x_2 \Rightarrow$ $\underline{(-1, 1)}$

5) Pokud $\{a_1, a_2, a_3\}$ je baze v V a $(1, 1, 1)$ jsou souř. vektora

a, jaky budou souř. a v Baze

$a_1 + a_2, a_1 + a_3, a_2 + a_3$?

$a = x_1(a_1+a_2) + x_2(a_1+a_3) +$

$+ x_3(a_2+a_3) =$

$= (x_1+x_2)a_1 + (x_1+x_3)a_2 + (x_2+x_3)a_3$

$$\Rightarrow \begin{cases} x_1+x_2 = 1 \\ x_1+x_3 = 1 \\ x_2+x_3 = 1 \end{cases} \Rightarrow \underline{x_1 = x_2 = x_3 = \frac{1}{2}}$$

6) V je prostor všech matic 2×2: součet prvku ve všech řadcich je nulový.

Najít Baze a dim V.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

V je podprostor v U: $\{\text{matice } 2\times 2\}$

Baze U: $\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$

$\dim U = 4$

$A = a_{11}E_1 + \dots + a_{22}E_4$

V je dan soustavem

z 2 rovnic: $\begin{cases} a_{11}+a_{12}=0 \\ a_{21}+a_{22}=0 \end{cases}$

$\begin{pmatrix}1&1&0&0\\0&0&1&1\end{pmatrix}$ - už schodi tvar

a_{12}, a_{22} - volný

$\Rightarrow (-1,1,0,0) \sim \begin{pmatrix}-1&1\\0&0\end{pmatrix}$

$(0,0,-1,1) \sim \begin{pmatrix}0&0\\-1&1\end{pmatrix}$

$\Big\}$ Baze

$\dim U = 2$

7) Ktery prostory maji konečne baze:

a) $V=\{f\in R[x]: f(x)=f(-x)$

b) $V=\{f: R\to R, f(-x)=f(x)$

c) $V=\{$posloupnosti real. čisel$\}$

d) $\{f(2x)=4f(x)\}\subset R[x]$?

a) $f(x)=f(-x)$

$\{1, x^2, x^4, \ldots, x^{2m}, \ldots\}$ -

- lin. nezavisl

$(a_0+a_2x^2+\ldots+a_{2m}x^{2m}=0)$

$\Rightarrow a_0 = \ldots = a_{2m} = 0$)

Baze je <u>max</u> l.n. nezav.

$\Rightarrow$ <u>nema</u>

B) V obsahuje prostor z přík. a) $\Rightarrow$ V ~~tak~~ taky <u>nema</u>

C) Sestrojime

$e_1 = (1, 0, 0, \ldots)$

$e_2 = (0, 1, 0, \ldots)$

a t d

$e_1, e_2, \ldots$ – l.n. nezav.

$\Rightarrow$ taky <u>nemuže</u> být

→ tedy nemůže být koneč. baze

d) $f(2x) = 4f(x)$

$a_0 + 2a_1x + 4a_2x^2 + 8a_3x^3 + \dots$

$= 4a_0 + 4a_1x + 4a_2x^2 + 4a_3x^3 + \dots$

$\Rightarrow a_0 = a_1 = a_3 = a_4 = \dots = 0$

$\Rightarrow V = [x^2]$ – ano!