3.7 f, g rostoucí na $I \subseteq D(f) \cap D(g)$

(i) $h(x) = f(x) + g(x)$

- $h(x)$ je rostoucí
- Dk: $x_1, x_2 \in I$, $x_1 < x_2$

chceme: $h(x_1) < h(x_2)$

$\Leftrightarrow$

$f(x_1) + g(x_1) < f(x_2) + g(x_2)$

$f(x_1) < f(x_2)$

$g(x_1) < g(x_2)$

$\underbrace{f(x_1) + g(x_1)}_{h(x_1)} < \underbrace{f(x_2) + g(x_2)}_{h(x_2)}$ $\square$

(ii) $h(x) = f(x) - g(x)$

$\rightarrow$ nemusí být rostoucí, klesající

- $f(x) = x$, $g(x) = 3x$ $\Big\}$ $h(x) = -2x$ klesající
- $f(x) = 3x$, $g(x) = x$ $\Big\}$ $h(x) = 2x$ rostoucí

(iii) $h(x) = f(x) \cdot g(x)$ $\qquad x_1, x_2 \in I$

? rostoucí ? $\qquad x_1 < x_2$

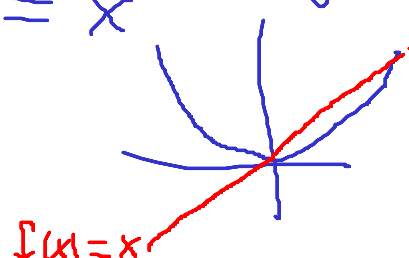
$$\cancel{\Downarrow}\left\{\begin{array}{l} f(x_1) < f(x_2) \\ g(x_1) < g(x_2) \end{array}\right.$$

$$f(x_1)g(x_1) < f(x_2)g(x_2)$$

Implikace platí např. pro f, g kladné funkce, ale nikoli obecně.

Př.: $f(x) = g(x) = x$ $\qquad$ rostoucí na $\mathbb{R}$

$h(x) = x^2$



(iv) $h(x) = -g(x)$

↳ klesající

$x_1 < x_2 \qquad x_1, x_2 \in I$

$g(x_1) < g(x_2) \quad / \cdot (-1)$

$h(x_1) = -g(x_1) > -g(x_2) = h(x_2)$

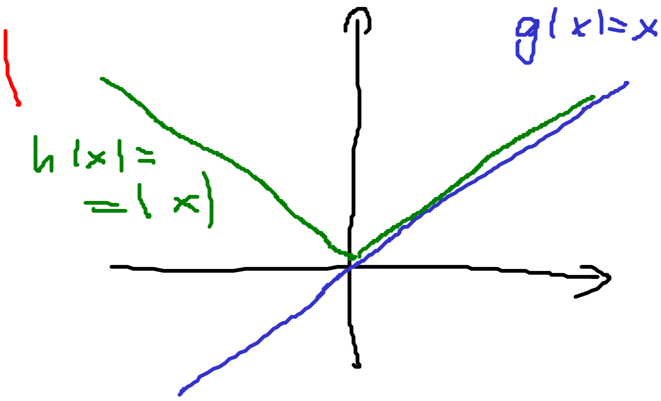
(v) $h(x) = g(x) \cdot g(x)$

$\rightarrow$ viz (iii)

(vi) $h(x) = |g(x)|$

obecně není rostoucí

$g(x) = x$ na $[-1,1]$



(vii) $h(x) = \frac{1}{g(x)}$

$g(x) \neq 0$ pro $x \in I$

- $g(x)$ kladná pro $x \in I$

$x_1 < x_2 \qquad x_1, x_2 \in I$

$g(x_1) < g(x_2) \quad / \frac{1}{g(x_1) \cdot g(x_2)}$

$\frac{1}{g(x_2)} < \frac{1}{g(x_1)}$

$h(x_2) < h(x_1)$

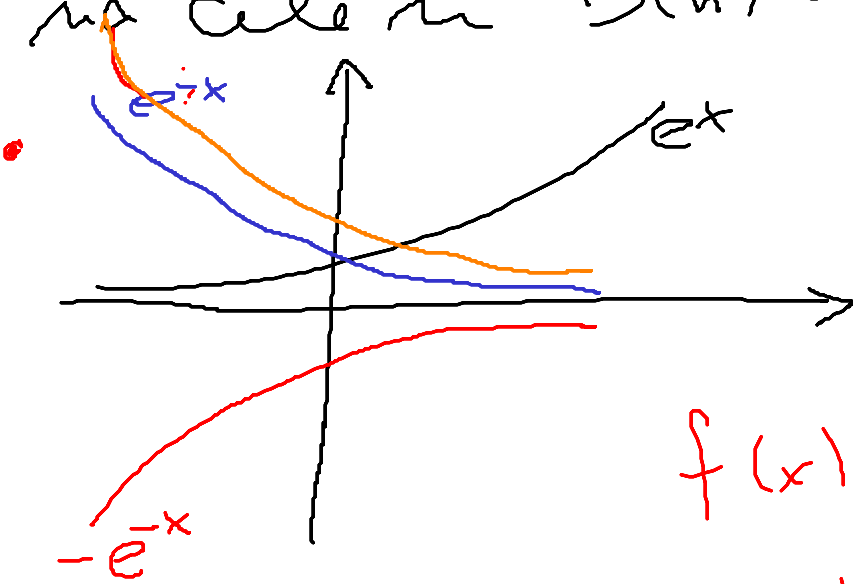
$h(x_1) > h(x_2)$ $\qquad$ h klesající

- $g(x)$ záporné pro $x \in I$

. . . . klesající

Závěr: $h(x)$ klesající na I

4.1 Dejte příklad rostoucích funkcí f, g s def. oborem $\mathbb{R}$ takových, že funkce $h(x) = f(x) \cdot g(x)$ je klesající na celém $D(h) = \mathbb{R}$



$f(x)$

$g(x)$

$f(x) = g(x) = -e^{-x}$

$h(x) = e^{-2x}$

$$h(x) = (-e^{-x}) \cdot (-e^{-x}) = e^{-2x}$$

4.2 • f rostoucí na $D(f) = \mathbb{R}$
má obor hodnot $H(f) = (0, \infty)$

• $g(x) = x \cdot f(x)$ je rostoucí na $(0, \infty)$. Dokažte.

$x_1 < x_2$ $\qquad$ $x_1, x_2 \in (0, \infty)$

$\underline{f(x_1) < f(x_2)}$ $\qquad$ $f(x_1), f(x_2) \in (0, \infty)$

$x_1 \cdot f(x_1) < x_2 \cdot f(x_2)$

$h(x_1) < h(x_2)$ $\square$

4.3 $\boxed{ax^2+bx+c=0}$, $a \neq 0$

$$a\left(x^2+\frac{b}{a}x\right)+c=0$$

$$a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right]=0$$

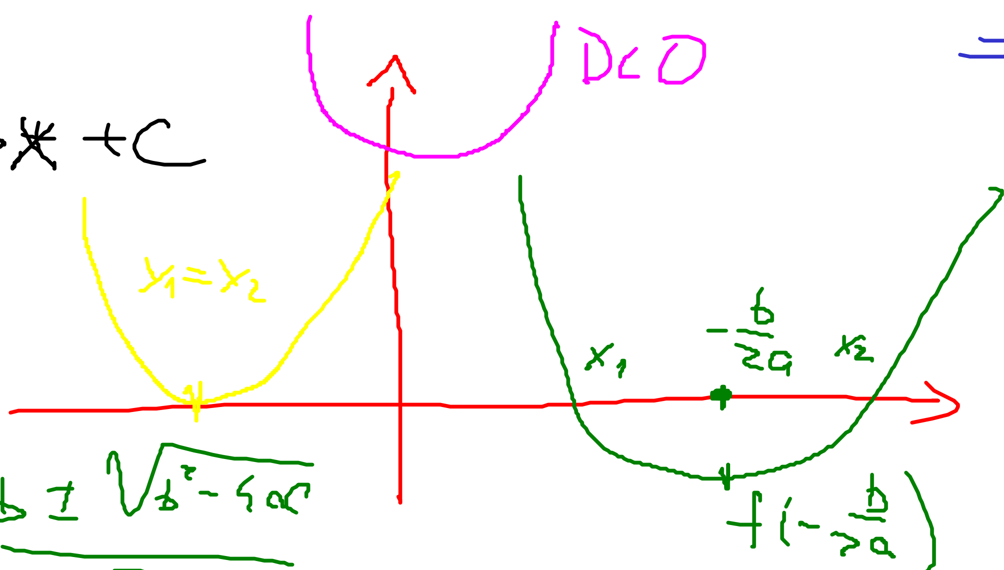
$$a\left[\left(x+\frac{b}{2a}\right)^2-\underbrace{\frac{b^2-4ca}{4a^2}}_{\left(\sqrt{\frac{b^2-4ac}{4a^2}}\right)^2}\right]=0$$

$$a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)$$

$$a\left[x-\left(\frac{-b-\sqrt{b^2-4ac}}{2}\right)\right]\cdot\left[x-\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\right]=0$$

$f(x)=ax^2+bx+c$

- $a>0$



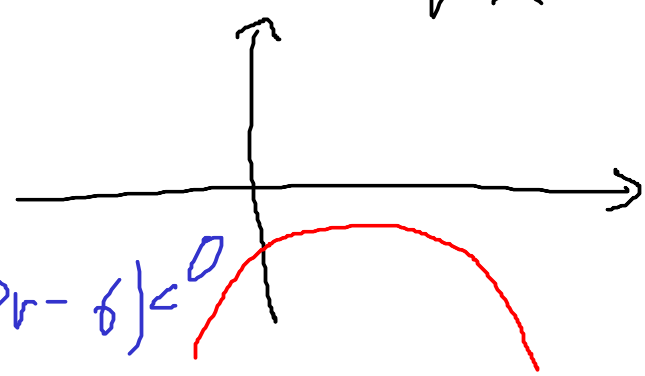
$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\frac{x_1+x_2}{2} = -\frac{b}{2a} \qquad D = b^2-4ac$$

$\cdot\ a<0$

4.4 $\underbrace{\underbrace{(r+4)}_{<0}x^2-2rx+2r-6}_{f(x)}<0$ $r \sim 0$ $\forall x\in\mathbb{R}$

$$\boxed{r<-4}$$



$$D=(-2r)^2-4(r+4)(2r-6)<0$$

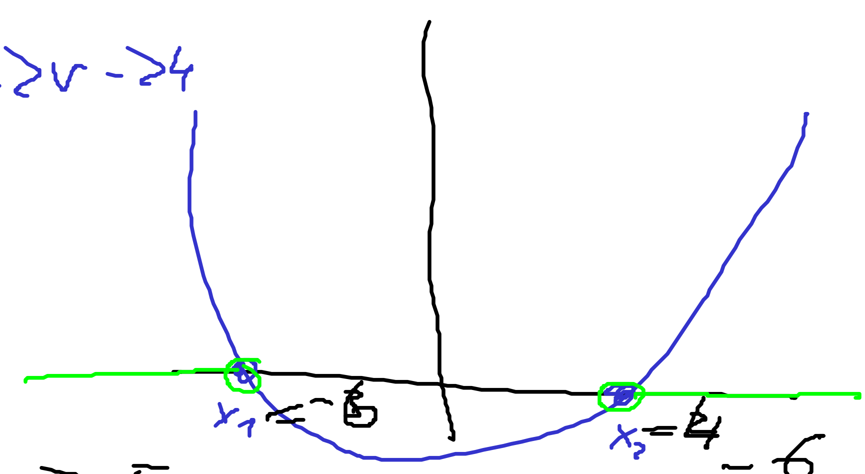
$$D=4r^2-4(2r^2-6r+8r-24)<0 \quad /\tfrac{1}{4}$$

$$r^2-(2r^2+2r-24)<0$$

$$-r^2-2r+24<0 \quad /\cdot(-1)$$

$$\boxed{r^2+2r-24>0}$$

$r^2+2r-24$



$$r^2+2r-24=0$$

$$r_{1,2}=\frac{-2\pm\sqrt{4+4\cdot 24}}{2}=\frac{-2\pm2\sqrt{1+24}}{2}=$$

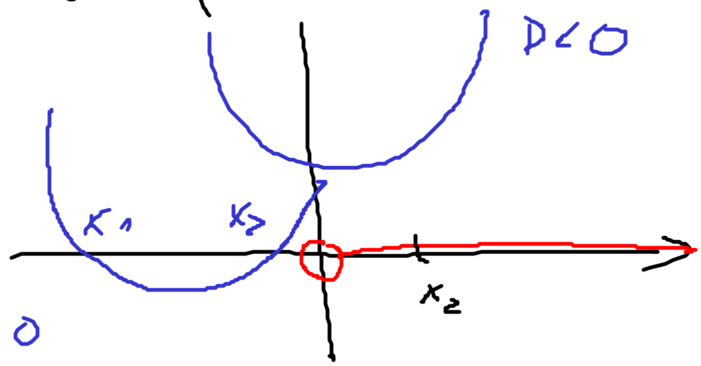
$$=\frac{-2\pm2\cdot5}{2}=-1\pm5=\begin{cases}-6\\4\end{cases}$$

Záměna: • $v < -4$

• $v \in (-\infty, -6) \cup (4, \infty)$

Tedy $v \in (-\infty, -6)$

(b) $\underbrace{v}_{>0}\underbrace{x^2 - 4x + 3v + 1}_{f(x)} > 0$ pro $x \in (0, \infty)$



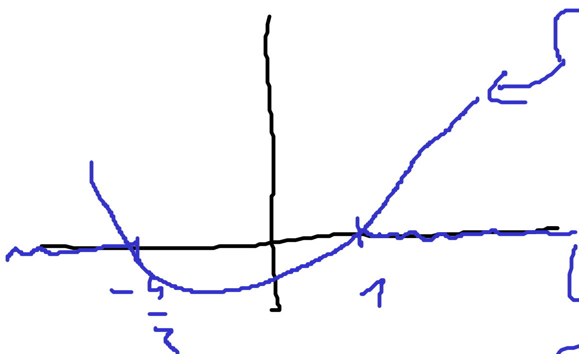
• $D < 0$

• $D \geq 0$ ⇝ jeden nebo 2 reálné kořeny

→ tyto kořeny ≤ 0

• $D = 16 - 4 \cdot v \cdot (3v+1) = -12v^2 - 4v + 16 < 0$ $\quad / \cdot \left(-\frac{1}{4}\right)$

$$\boxed{\underbrace{3}_{>0}v^2 + v - 4 > 0}$$



$$v_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \cdot 3 \cdot (-4)}}{6} = \frac{-1 \pm 7}{6} = \begin{cases} 1 \\ -\frac{8}{6} = -\frac{4}{3} \end{cases}$$

platí

$v \in \left(-\infty, -\frac{4}{3}\right) \cup (1, \infty)$

Jelikož $v > 0$, dostaneme $\boxed{v \in (1, \infty)}$

výsledek

· Reálné kořeny jsou:

$$f(x) = r x^2 - 4x + 3r + 1 = 0$$

$$r_{1,2} = \frac{4 \pm \sqrt{4(-3r^2 - r + 4)}}{2r} \leq 0$$

$$4 \pm 2\sqrt{(-3r^2 - r + 4)} \leq 0$$

$$4 + 2\sqrt{\ldots\ldots} \leq 0$$

nejde

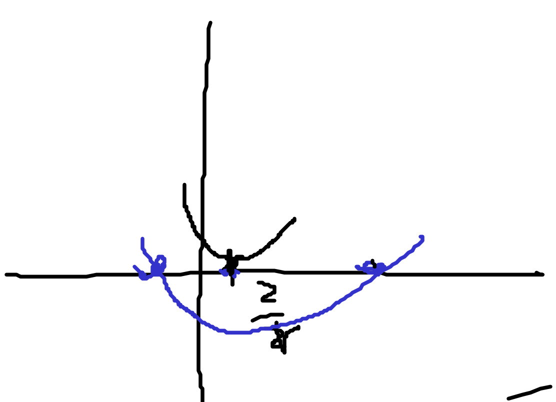
Jiná úvaha: $\forall x > 0$

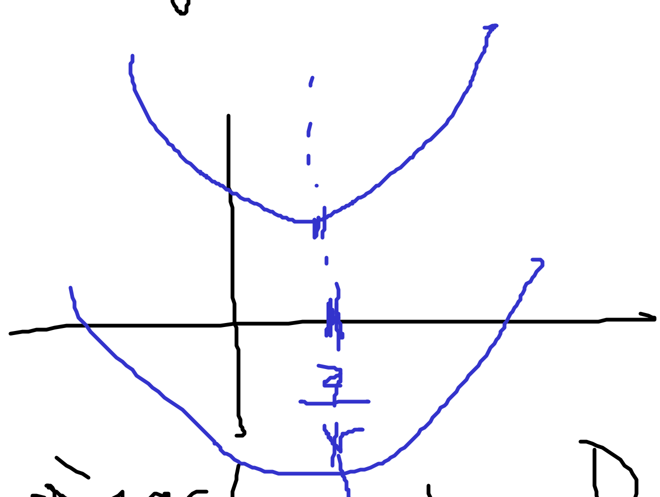
$$r x^2 - 4x + 3r + 1 > 0 \quad , \quad r > 0$$

$$r\left(x^2 - \frac{4}{r}x\right) + 3r + 1 > 0$$

$$r\left[\left(x - \frac{2}{r}\right)^2 - \frac{4}{r^2}\right] + 3r + 1 > 0$$

$$r\left(x - \frac{2}{r}\right)^2 + 3r - \frac{4}{r} + 1 > 0$$



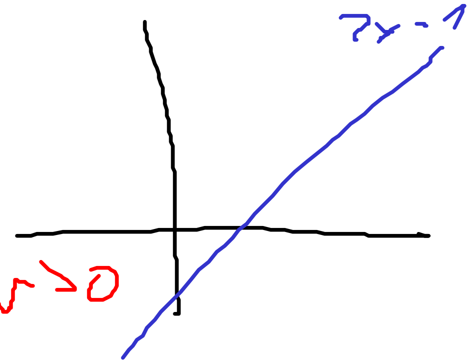


=> jediná možnost je $D < 0$

Bez reálných kořenů

(c) $(r-2)x^2 + rx + 1 - r > 0 \quad \forall x > 0$

$\underbrace{(r-2)}_{r>2}$ $\qquad$ $\underline{r=2}$: $2x - 1 > 0$



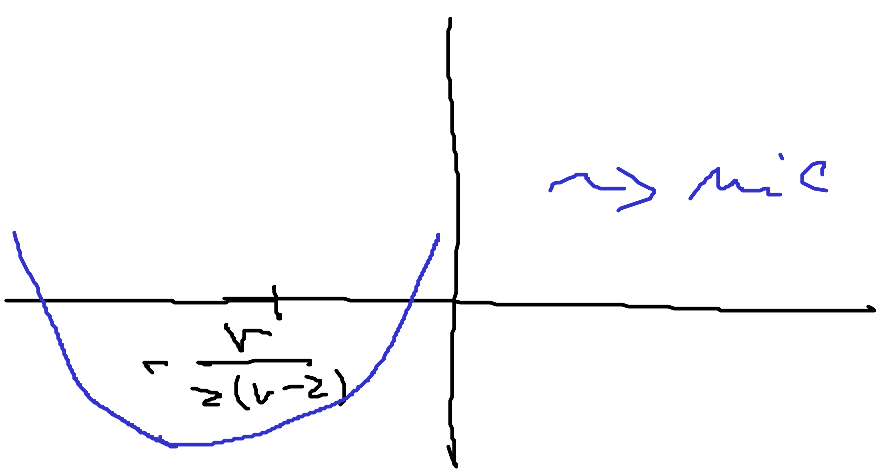
$$(r-2)\left(x^2 + \frac{r}{r-2}x\right) + 1 - r > 0$$

$$(r-2)\left[\left(x + \frac{r}{2(r-2)}\right)^2 - \frac{r^2}{4(r-2)^2}\right] + 1 - r > 0$$

$$(r-2)\left(\underbrace{x + \frac{r}{2(r-2)}}\right)^2 - \frac{r^2}{4(r-2)} + 1 - r > 0$$

střed paraboly je v bodě $-\frac{r}{2(r-2)} < 0$

- $D < 0$
- jsou reálné kořeny ≤ 0



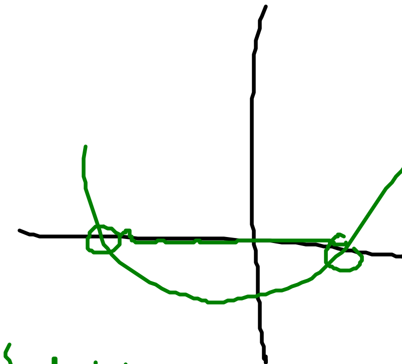
$\rightsquigarrow$ nic nového

$$D = v^2 - 4(v-2)(1-v) < 0 \qquad v > 2$$

$$v^2 - 4(-v^2 + v + 2v - 2) < 0$$

$$5v^2 - 12v + 8 < 0$$

$$v_{1,2} = \frac{12 \pm \sqrt{144 - 20 \cdot 8}}{10}$$



nemá reálné kořeny

$$\Rightarrow D > 0$$

$$x_{1,2} = \frac{-v \pm \sqrt{5v^2 - 12v + 8}}{2(v-2)} \leq 0$$

$$-v + \sqrt{5v^2 - 12v + 8} \leq 0$$

$$\sqrt{5v^2 - 12v + 8} \leq v \quad /()^2$$

$$5v^2 - 12v + 8 \leq v^2$$

$$4v^2 - 12v + 8 \leq 0$$

$$v^2 - 3v + 2 \leq 0$$

$$v_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} \quad v \in (1,2)$$

závěr

pro žádné v to neplatí

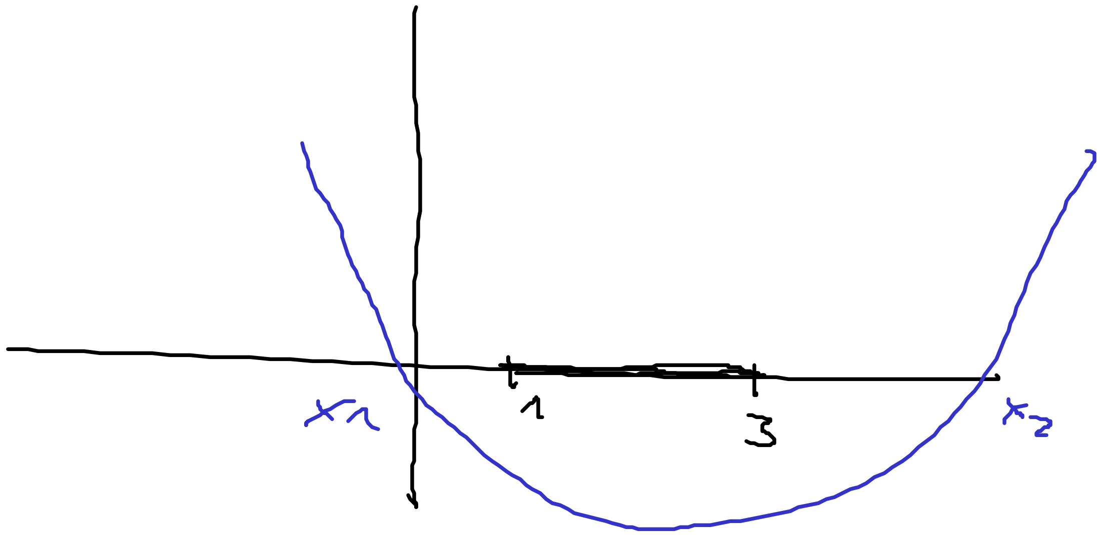
(d) $(x-3v)(x-v-3) < 0$ pro $x \in [1,3]$

$x_1 = 3v$

$x_2 = v+3$

$x^2 + ---- < 0$

↳ parabola otočená nahoru



• $3v < 1 \wedge 3 < v+3$

$v < \frac{1}{3} \wedge v > 0 \Rightarrow v \in (0, \frac{1}{3})$

• $v+3 < 1 \wedge 3 < 3v$

$v < -2 \wedge 1 < v$ nejde

Závěr: $v \in (0, \frac{1}{3})$