2. vnitrosemestrální test

Př 2: Kvadratické soustavy

Určete všechny parametry $a \in \mathbb{R}$ t. ž. následující soustava má <u>reálné</u> řešení:

$$\boxed{\begin{aligned} a x^2 + x + a &= 0 \\ x^2 + a x + a &= 0 \end{aligned}} \quad / \cdot (-a)$$

Řeš.:

$$x + a - a(ax + a) = 0$$

$$(1 - a^2)x + a - a^2 = 0$$

$$\boxed{(1+a)(1-a)\,x + a(1-a) = 0} \Leftarrow$$

nepl.

(1) $a = 1$

(2) $a \neq 1$, $a \neq -1$ $\Rightarrow$ $x = -\dfrac{a}{1+a}$ $\Leftarrow$ ($a = -1$)

(1) $a = 1$ $\rightsquigarrow$ $x^2 + x + 1 = 0$

$D = 1 - 4 = -3 < 0$

$\times$

pouze kompl. řešení

2) $x = -\frac{a}{1+a}$ ⇝ do 2. rovnice

$$\left(-\frac{a}{1+a}\right)^2 + a\cdot\left(-\frac{a}{1+a}\right) + a = 0 \quad /(1+a)^2$$

$$a^2 - a^2(1+a) + a(1+a)^2 = 0$$

- $a = 0$
- $a \neq 0$ ⇝ $a - a(1+a) + (1+a)^2 = 0$

$$-a^2 + (a^2 + 2a + 1) = 0$$

$$2a + 1 = 0$$

$$a = -\frac{1}{2}$$

Dosadíme:

- $a = 0$ ⇝ $x = 0$ ✓
- $a = -\frac{1}{2}$ ⇝ $-\frac{1}{2}x^2 + x - \frac{1}{2} = 0 \quad /\cdot(-2)$

$\downarrow$

$x = 1$

$$x^2 - \frac{1}{2}x - \frac{1}{2} = 0$$

$$x^2 - 2x + 1 = 0$$ ⇝ $x = 1$

společné řešení

Závěr: $a = 0$

$a = \frac{1}{2}$

Alternativní:

$$ax^2 + x + a = 0$$

$$x^2 + ax + a = 0$$

Odečtem: $(a-1)x^2 + (1-a)x = 0$

$$(a-1)x(x-1) = 0$$

$a = 1 \;\to\;$ nejde

$x = 0 \;\to\; a = 0$

$x = 1 \;\to\; a = -\frac{3}{2}$

Př 3: odmocniny

$$\underbrace{\sqrt{x+2} - \sqrt{x-1}}_{>0} > \underbrace{\sqrt{2x-3}}_{\geq 0} \quad /()^2$$

$$\sqrt{x+2} > \sqrt{2x-3} + \sqrt{x-1} \quad /()^2$$

$x > -2$ $\quad x \geq \frac{3}{2}$ $\quad x \geq 1$

$$x+2 > (2x-3) + (x-1) + 2\sqrt{(2x-3)(x-1)}$$

$$\underbrace{-2x+6} > 2\sqrt{(2x-3)(x-1)} \quad /()^2$$

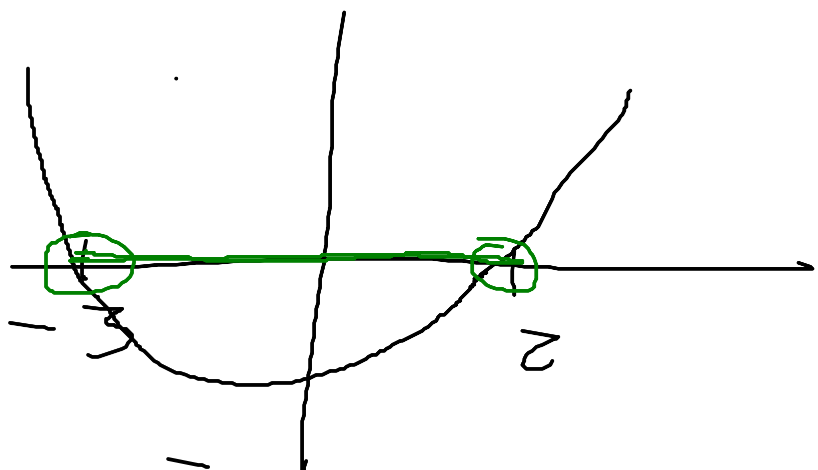
$> D \Rightarrow x < 3$

$(x+3)^2 > 2x^2 - 2x - 3x + 3$

$x^2 - 6x + 9$

$0 > x^2 + x - 6 = (x+3)(x-2)$

$x \in (-3, 2)$



Zonie : $x \in [\frac{3}{2}, 2)$

Pr 4: (a) $\log_3(x^2) \cdot \log_9 x = 3$

$2 \log_3 x \cdot \frac{\log_3 x}{\log_3 9} = 3$

$(\log_3 x)^2 = 3$

$$\log_3 x = \pm\sqrt{3} \quad / \; 3^{(\cdot\cdot)}$$

$$x = 3^{\pm\sqrt{3}}$$

(b) $\log_3 (\underbrace{|n|+1}_{x})^2 \cdot \log_9 (\underbrace{|n|+1}_{x}) = 3$

$$|n| + 1 = 3^{\pm\sqrt{3}}$$

$$|n| = \begin{cases} 3^{\sqrt{3}} - 1 \\ 3^{-\sqrt{3}} - 1 = \dfrac{1}{3^{\sqrt{3}}} - 1 < 0 \end{cases}$$

Závěr:

$$\boxed{n = \pm\left(3^{\sqrt{3}} - 1\right)}$$

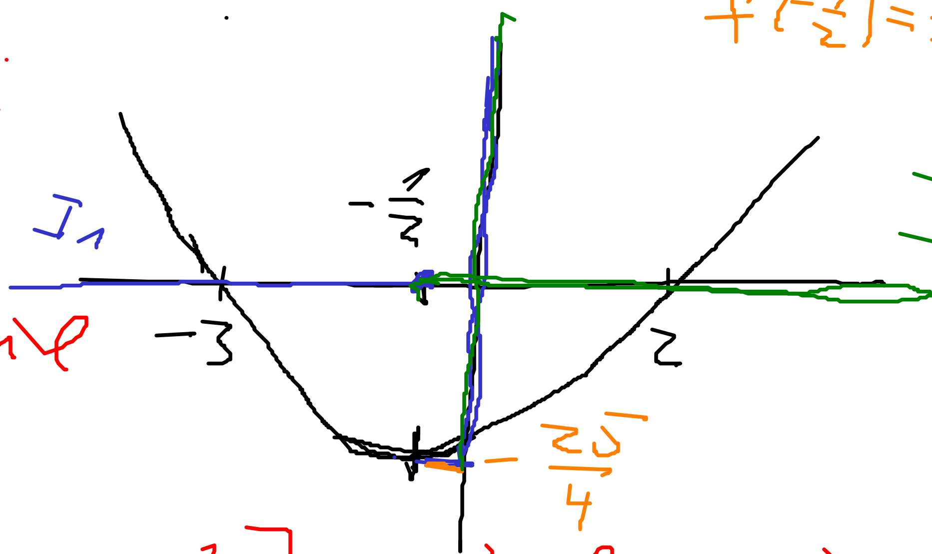
$$3^{\sqrt{3}} > 1$$

$$\frac{1}{3^{\sqrt{3}}} < 1$$

10.1 Určete intervaly kde je $f(x)$ monotónní k inverzní funkce.

(1) $f(x) = x^2 + x - 6 = (x+3)(x-2)$

$f(-\frac{1}{2}) = \frac{5}{2}\cdot(-\frac{5}{2})$



maximální intervaly monotonie jsou

$I_1 = (-\infty, -\frac{1}{2}]$ klesající

$I_2 = [-\frac{1}{2}, \infty)$ rostoucí

$$y = x^2 + x - 6$$

$$0 = x^2 + x - 6 - y$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1 + 4(y+6)}}{2} = \frac{-1 \pm \sqrt{4y + 25}}{2}$$

$$= -\frac{1}{2} \pm \sqrt{y + \frac{25}{4}}$$

Označíme $h(x) = f^{-1}(x)$

inverzní funkci

Interval $I_1 = (-\infty, -\frac{1}{2}]$

$$h(y) = -\frac{1}{2} - \sqrt{y + \frac{25}{4}}$$

$$D(h) = [-\frac{25}{4}, \infty)$$

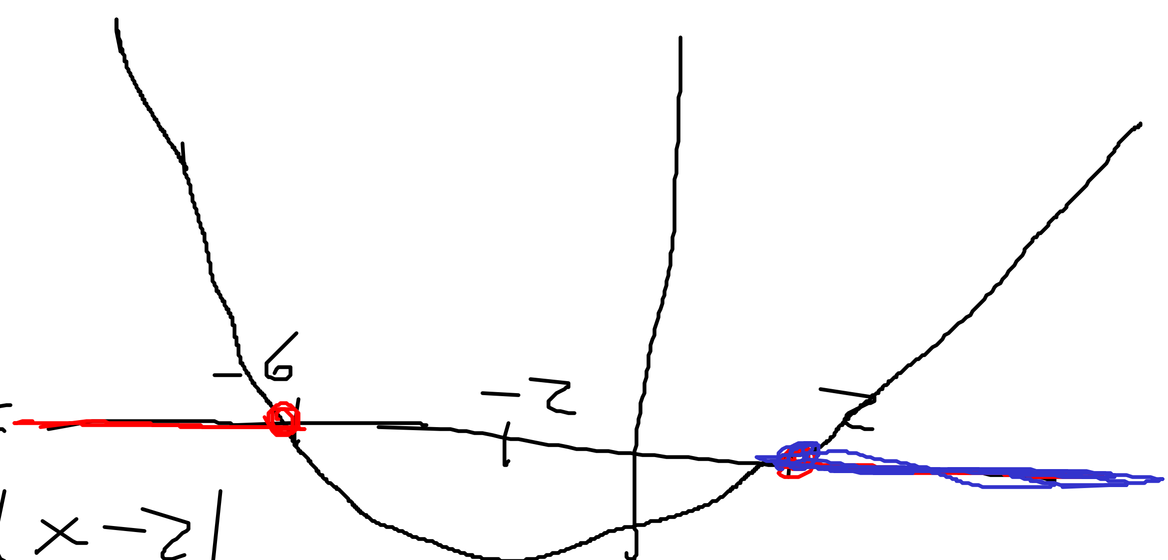
$$H(h) = (-\infty, -\frac{1}{2}]$$

Interval $I_2 = [-\frac{1}{2}, \infty)$

podobně

② $f(x) = \sqrt{x^2 + 4x - 12}$

$x^2 + 4x - 12 =$



$= (x+6)(x-2)$

$$D(f) = \underbrace{(-\infty, -6]}_{I_1} \cup \underbrace{[2, \infty)}_{I_2}$$

$f(x)$ klesající rostoucí

Inverzní funkce $h(y) = f^{-1}(y)$

$$y = \sqrt{x^2 + 4x - 12}, \quad y \geq 0$$

$$y^2 = x^2 + 4x - 12$$

$$0 = x^2 + 4x - y^2 - 12$$

$$x_{1,2} = \frac{-4 \pm \sqrt{16 + 4(y^2 + 12)}}{2} =$$

$$= -2 \pm \sqrt{4 + (y^2 + 12)}$$

$$= -2 \pm \sqrt{y^2 + 16}$$

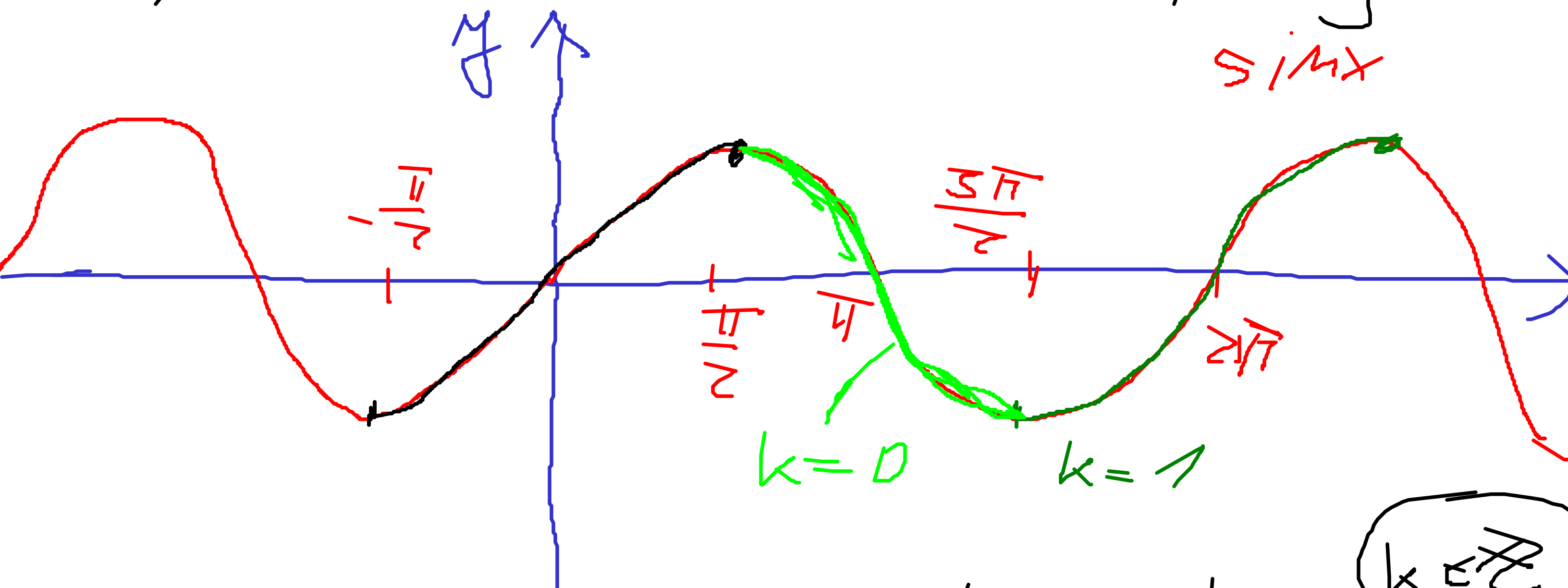
$I_1 = (-\infty, -6]$: $h(y) = -2 - \sqrt{y^2 + 16}$

definovaná na $D(h) = [0, \infty)$

$H(h) = (-\infty, -6]$

$I_2 = [2, \infty)$

10.2: arcsin je inverzní fukce k funkci $\sin x$ na intervalu $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$k \in \mathbb{Z}$

(a) $\sin x$ na intervalu $x \in \left[2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}\right]$

$h(y)$ – hledaná inverzní funkce

$y = \sin x = \sin(\underbrace{x - 2k\pi}_{\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]})$ /arcsin

$$\arcsin y = x - 2k\pi$$

$$x = \arcsin y + 2k\pi$$

Závěr: $h(y) = \arcsin y + 2k\pi$

(b) $\sin x$ na intervaly

$$\left[(2k+1)\pi - \frac{\pi}{2},\ (2k+1)\pi + \frac{\pi}{2}\right]$$

$$y = \sin x = \sin(\underbrace{x - 2k\pi}_{\in\left[\frac{\pi}{2},\ \frac{3\pi}{2}\right]}) =$$

$$\boxed{\begin{aligned}\sin x &= \sin(\pi - x)\\ &= -\sin(x - \pi)\end{aligned}}$$

$$= -\sin(x - 2k\pi - \pi)$$

$$= -\sin(\underbrace{x - (2k+1)\pi}_{\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]})$$

$$y = \sin x(-x + (2k+1)\pi) \,/\, \arcsin$$

$$\arcsin y = -x + (2k+1)\pi$$

$x = -\arcsin y + (2k+1)\pi$

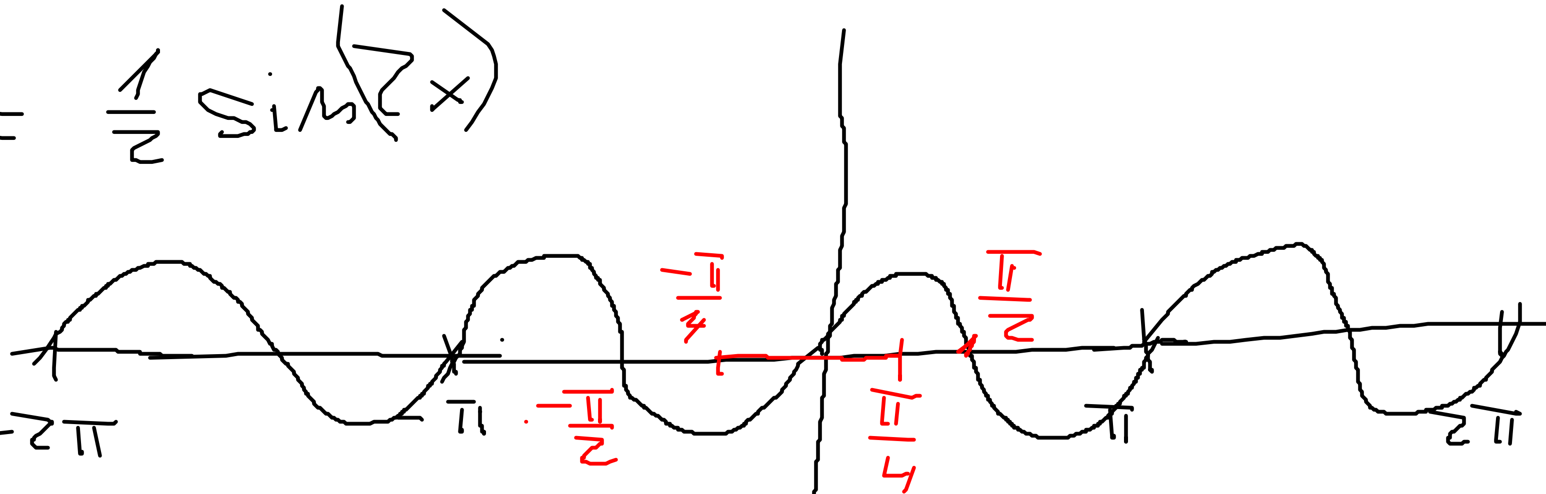
$h(y) = -\arcsin y + (2k+1)\pi$

10.3 Určete max interval monotonie funkce $f(x)$ obsahující 0 a pak inverzní funkci.

(1) $f(x) = \sin x \cdot \cos x =$

$\sin 2x = 2 \sin x \cos x$

$= \frac{1}{2} \sin(2x)$



$I = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ rostoucí

$y = f(x) = \frac{1}{2} \sin(2x), \; x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$2y = \sin 2x \quad \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arcsin(2y) = 2x$

závěr: inverzní funkce

$$h(y) = \frac{1}{2}\arcsin(2y)$$

(2) $f(x) = \sin x + \cos x =$

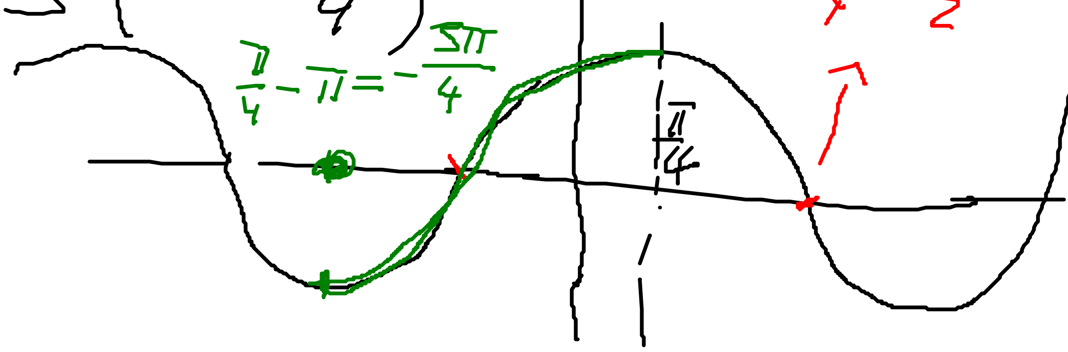
$$\boxed{\begin{array}{l} \text{Minule } \cos x = \sin\left(\frac{\pi}{2} - x\right) \\ \sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \end{array}}$$

$$= \sin x + \sin\left(\frac{\pi}{2} - x\right) =$$

$$= 2 \sin \frac{x + \left(\frac{\pi}{2} - x\right)}{2} \cos \frac{x - \left(\frac{\pi}{2} - x\right)}{2}$$

$$= 2 \sin \frac{\pi}{4} \cos\left(\frac{2x - \frac{\pi}{2}}{2}\right)$$

$$= \sqrt{2} \cos\left(x - \frac{\pi}{4}\right)$$



$\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$

$$\boxed{y = \sqrt{2}\cos(\underbrace{x - \tfrac{\pi}{4}}_{\in[-\pi,0]})}$$

nema inverz "arccos"

$$\frac{1}{\sqrt{2}}\, y = \cos\left(x - \frac{\pi}{4}\right)$$

$$\arccos\left(\frac{y}{\sqrt{2}}\right) = x - \frac{\pi}{4}$$

~~zove~~: $x = h(y) = \text{"}\arccos\text{"}\left(\frac{y}{\sqrt{2}}\right) + \frac{\pi}{4}$

Spremno: arccos je inverz $\cos x$ na $[0, \pi]$

$$y = \sqrt{2}\cos(\underbrace{x - \tfrac{\pi}{4}}_{\in[-\pi,0]}) =$$

$$= \sqrt{2}\cos(\underbrace{-x + \tfrac{\pi}{4}}_{\in[0,\pi]})$$

$$\frac{y}{\sqrt{2}} = \cos\left(-x + \frac{\pi}{4}\right)$$

$$\arccos\left(\frac{y}{\sqrt{2}}\right) = -x + \frac{\pi}{4}$$

$$x = \underbrace{-\arccos\left(\frac{y}{\sqrt{2}}\right) + \frac{\pi}{4}}_{h(y)}$$