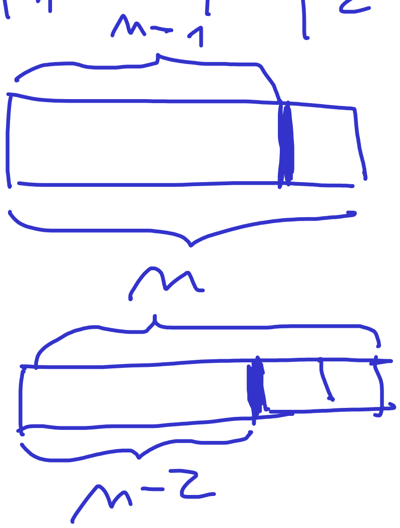
12.1 Kolika způsoby lze vyjít 10 schodů pokud lze dělat pouze kroky o 1 nebo 2 schody?

P_n – počet způsobů, jak za těchto podmínek vyjít n schodů

$P_1 = 1$, $P_2 = 2$, $P_3 = 1 + 2 = 3$



$$P_n = P_{n-1} + P_{n-2}$$

Fibonacciho posloupnost

$P_4 = P_3 + P_2 = 3 + 2 = 5$

$P_5 = P_4 + P_3 = 5 + 3 = 8$

$P_6 = 8 + 5 = 13$

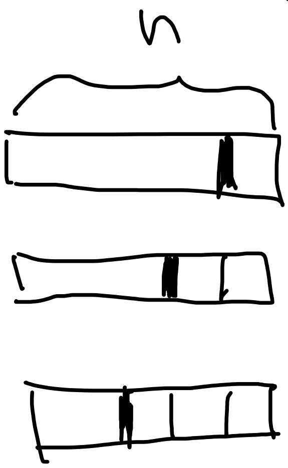
$P_7 = 13 + 8 = 21$

$P_8 = 21 + 13 = 34$

$P_9 = 34 + 21 = 55$

$P_{10} = 55 + 34 = \underline{\underline{89}}$

② kroky o 1 nebo 2 nebo 3 schody



$P_n = P_{n-1} + P_{n-2} + P_{n-3}$

$P_1 = 1, P_2 = 2, P_3 = 4$

$P_4 = 4 + 2 + 1 = 7$

$P_5 = 7 + 4 + 2 = 13$

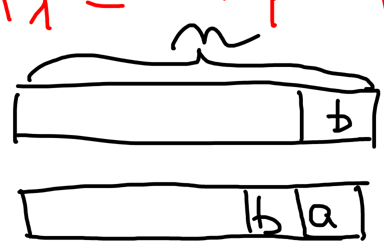
$P_6 = 13 + 7 + 4 = 24$

......

12.2 ① Určete, kolik je řetězců délky $n = 10$ nad abecedou $\{a, b\}$, které neobsahují podřetězec aa.

P_n – počet řetězců délky n neobsahujících podřetězec aa

$P_1 = 2, \quad P_2 = 3$

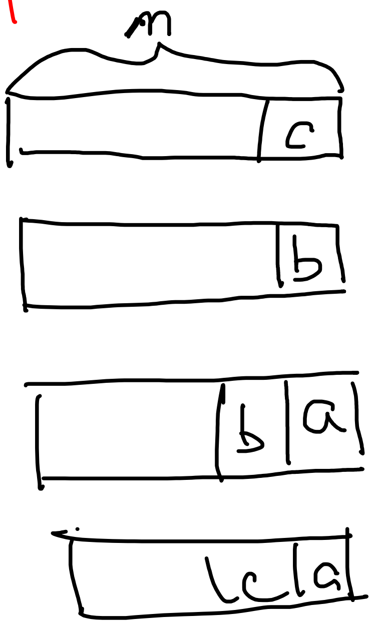


$P_n = P_{n-1} + P_{n-2}$

P_{10}
$\parallel$
2, 3, 5, 8, 13, 21, 34, 55, 89, 144

(2) abeceda $\{a, b, c\}$, neobsahuje aa.

$P_1 = 3, \; P_2 = 8$



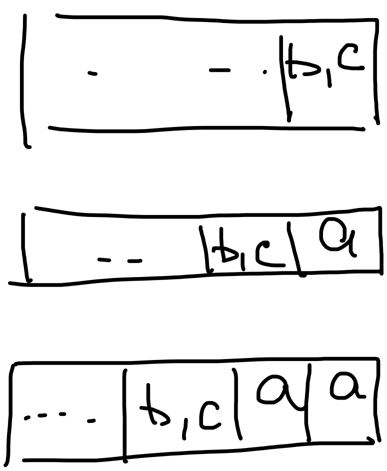
$$P_n = P_{n-1} + P_{n-1} + P_{n-2} + P_{n-2}$$

$$= 2P_{n-1} + 2P_{n-2}$$

$3, 8, 22, 60, 164, 2 \cdot 234 = 468,$

$\dots$

(3) abeceda $\{a, b, c\}$, neobsahuje podřetězec aaa



$$P_n = 2P_{n-1} + 2P_{n-2} + 2P_{n-3}$$

$P_1 = 3, \; P_2 = 9,$

$P_3 = 27 - 1 = 26$

12.3: $P_n = P_{n-1} + P_{n-2}$

$P_1 = 1, \; P_2 = 2$

$$\boxed{p_n - p_{n-1} - p_{n-2} = 0} \quad (*)$$

$\rightsquigarrow$ zkusmo najít p_n v exponenciální tvaru

$p_n = \lambda^n$ $\quad \lambda \neq 0$

$$\lambda^n - \lambda^{n-1} - \lambda^{n-2} = \underbrace{\lambda^{n-2}}_{\neq 0}\underbrace{(\lambda^2-\lambda-1)}_{\Downarrow} = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$$

$$p_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$p_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

dvě možná řešení rovnice $(*)$

- $p_n, \bar{p}_n$ vyhovují rovnici

$$p_n - p_{n-1} - p_{n-2} = 0$$

Pak i $a p_n + \bar{a}\bar{p}_n$, $\quad a, \bar{a} \in \mathbb{R}$

vyhovují této rovnici.

Obecné řešení rovnice (*) je tvaru

$$\boxed{p_n = A\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n + B\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad A, B \in \mathbb{R}}$$

$$p_1 = A\,\frac{1+\sqrt{5}}{2} + B\,\frac{1-\sqrt{5}}{2} = 1$$

$$p_2 = A\cdot\frac{1+2\sqrt{5}+5}{4} + B\,\frac{1-2\sqrt{5}+5}{4} = 2$$

$$\to A(1+\sqrt{5}) + B(1-\sqrt{5}) = 2 \quad / -(1+\sqrt{5})$$

$$A(1+2\sqrt{5}+5) + B(1-2\sqrt{5}+5) = 8$$

$$-B\underbrace{(1-\sqrt{5})(1+\sqrt{5})}_{1-5=-4} + B(1-2\sqrt{5}+5) = 6-2\sqrt{5}$$

$$B(4+1-2\sqrt{5}+5) = 6-2\sqrt{5}$$

$$B(10-2\sqrt{5}) = 6-2\sqrt{5}$$

$$B = \frac{3-\sqrt{5}}{5-\sqrt{5}}\cdot\frac{5+\sqrt{5}}{5+\sqrt{5}} = \frac{15+3\sqrt{5}-5\sqrt{5}-5}{5^2-5}$$

$$= \frac{10-2\sqrt{5}}{20} = \frac{5-\sqrt{5}}{10}$$

$$A(1+\sqrt{5}) + \frac{5-\sqrt{5}}{10}(1-\sqrt{5}) = 2$$

$$A(1+\sqrt{5}) + \frac{5-\sqrt{5}-5\sqrt{5}+5}{10} = 2$$

$$A(1+\sqrt{5}) + \frac{10-6\sqrt{5}}{10} = 2$$

$$A = \frac{\frac{-5+3\sqrt{5}}{5}+2}{1+\sqrt{5}} = \frac{-5+3\sqrt{5}+10}{5\cdot(1+\sqrt{5})}$$

$$A = \frac{5+3\sqrt{5}}{5(1+\sqrt{5})} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$= \frac{5-5\sqrt{5}+3\sqrt{5}-15}{5(1-5)} = \frac{-10-2\sqrt{5}}{-20}$$

$$A = \frac{5+\sqrt{5}}{10} \qquad B = \frac{5-\sqrt{5}}{10}$$

<u>Výsledek</u>: $\boxed{p_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n}$

p_n celočíselné $\forall n \in \mathbb{N}$