# KRIVKOVÉ INTEGRÁLY I. DRUHU

$I = \int_{\varphi} f\, ds$ ; $f$ je f-cia $n$-premenných definovaná na krivke $\varphi : [a,b] \longrightarrow \mathbb{R}^n$

$\leadsto$ $\varphi(t)$ je parametrizácia danej krivky

$\longrightarrow$ krivkový integrál pomocou parametrizácie prevedieme na určitý integrál :

$$I = \int_a^b f(\varphi(t)) \cdot \|\varphi'(t)\|\, dt$$ ; $\|\cdot\|$ je euklidovská norma v $\mathbb{R}^n$

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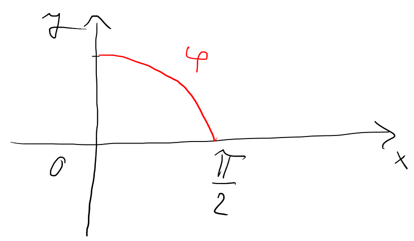
1. $I = \int_{\varphi} \sin 2x \, ds$ , $\varphi$ : graf f-cie $y = \cos x$ , $x \in [0, \frac{\pi}{2}]$

funkcia $f(x,y) = \sin 2x$ , krivka :




$\leadsto$ parametrizácia krivky :

$\varphi(t) = (x(t), y(t))$ , napríklad :

$x = t$ , $y = \cos t$ , $t \in [0, \frac{\pi}{2}]$

$\leadsto$ derivácia $\varphi'(t) = (x'(t), y'(t))$ .... $x'(t) = 1$ , $y'(t) = -\sin t$

$$\|\varphi'(t)\| = \sqrt{(x')^2 + (y')^2} = \sqrt{1 + \sin^2 t}$$

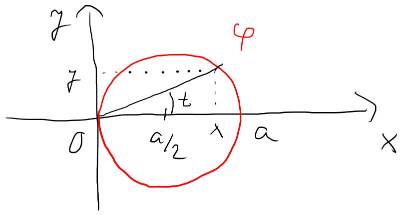
$$I = \int_0^{\pi/2} \sin 2t \cdot \sqrt{1 + \sin^2 t}\, dt = \left| \begin{array}{ll} u = 1 + \sin^2 t , & du = 2 \sin t \cos t\, dt \\ & \quad = \sin 2t\, dt \\ 0 \leadsto 1 , & \pi/2 \leadsto 2 \end{array} \right|$$

$$= \int_1^2 \sqrt{u}\, du = \left[ \frac{2}{3} u^{3/2} \right]_1^2 = \frac{2}{3} (\sqrt{8} - 1) = \boxed{\frac{2}{3} (2\sqrt{2} - 1)}$$

2.

$$I = \int_\varphi \sqrt{x^2+y^2}\, ds \quad ; \quad \varphi : x^2+y^2 = ax \quad ; \quad a>0$$

$f(x,y) = \sqrt{x^2+y^2}$ , $\varphi$ :




parametrizácia :

$\varrho = a\cos t$ , $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ; $x = \varrho \cos t = a\cos^2 t$

$y = \varrho \sin t = a \sin t \cos t = \frac{1}{2} a \sin 2t$

$$\varphi(t) = \left(a\cos^2 t \,,\, \frac{1}{2} a \sin 2t\right) \,,\, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\varphi'(t) = \left(-2a\cos t \sin t \,,\, a\cos 2t\right) = \left(-a\sin 2t \,,\, a\cos 2t\right)$$

$$\|\varphi'(t)\| = \sqrt{(-a\sin 2t)^2 + (a\cos 2t)^2} = a$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2\cos^4 t + a^2 \sin^2 t \cos^2 t}\; a\, dt = \int_{-\frac{\pi}{2}}^{\pi/2} a^2 \cdot \sqrt{\cos^2 t}\, dt = a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\, dt$$

$$= a^2 \left[\sin t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{2a^2}$$

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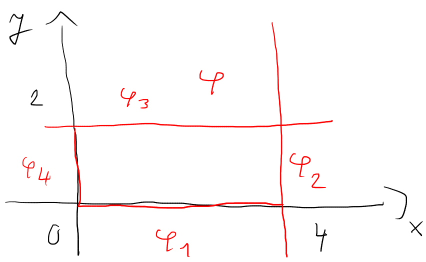
3.

$$I = \int_\varphi xy\, ds \quad , \quad \varphi : \text{obvod obdĺžnika} \quad x=0 \,,\, x=4$$

$$y=0 \,,\, y=2$$

$f(x,y) = xy$ , $\varphi$ :




$$\varphi = \varphi_1 \oplus \varphi_2 \oplus \varphi_3 \oplus \varphi_4$$

$$I = I_1 + I_2 + I_3 + I_4$$

a) $I_1 = \int_{\varphi_1} xy\, ds$ , $\varphi_1(t) = (t, 0)$ , $t \in [0,4]$

$\varphi_1'(t) = (1,0) \rightsquigarrow \|\varphi_1'(t)\| = 1$

$$I_1 = \int_0^4 t \cdot 0 \cdot 1\, dt = 0$$

b) $I_2 = \int_{\varphi_2} xy\, ds$ , $\varphi_2(t) = (4, t)$ , $t \in [0,2]$

$\varphi_2'(t) = (0,1) \rightsquigarrow \|\varphi_2'(t)\| = 1$

$$I_2 = \int_0^2 4t\, dt = \left[2t^2\right]_0^2 = 8$$

c) $I_3 = \int_{\varphi_3} xy\, ds$ , $\varphi_3(t) = (t, 2)$ , $t \in [0,4]$

$\varphi_3'(t) = (1,0) \rightsquigarrow \|\varphi_3'(t)\| = 1$

$$I_3 = \int_0^4 2t\, dt = \left[t^2\right]_0^4 = 16$$

d) $I_4 = \int_{\varphi_4} xy\, ds$ ; $\varphi_4(t) = (0, t)$ , $t \in [0,2]$

$\varphi_4'(t) = (0,1) \rightsquigarrow \|\varphi_4'(t)\| = 1$

$$I_4 = \int_0^2 0 \cdot t\, dt = 0$$

$$\Rightarrow I = 0 + 8 + 16 + 0 = \boxed{24}$$

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4.

$$I = \int_\varphi (x^2+y^2+z^2)\,ds \quad , \quad \varphi \text{ je skrutkovica :}$$

$$x = a\cos t \ , \ y = a\sin t \ , \ z = b\cdot t$$

$$t \in [0, 2\pi] \ , \ a, b > 0$$

$$\varphi(t) = (a\cos t \ , a\sin t \ , bt) \ , t \in [0, 2\pi]$$

$$\varphi'(t) = (-a\sin t \ , a\cos t \ , b) \rightsquigarrow \|\varphi'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= \sqrt{a^2+b^2}$$

$$I = \int_0^{2\pi} (a^2\cos^2 t + a^2\sin^2 t + b^2t^2)\cdot\sqrt{a^2+b^2}\,dt$$

$$= \int_0^{2\pi} (a^2 + b^2t^2)\sqrt{a^2+b^2}\,dt = \sqrt{a^2+b^2}\left[a^2t + \frac{b^2}{3}t^3\right]_0^{2\pi}$$

$$= \boxed{\sqrt{a^2+b^2}\left(2\pi a^2 + \frac{8\pi^3 b^2}{3}\right)}$$

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5.

$$I = \int_\varphi (x + y^2 - z)\,ds \quad , \quad \varphi : \text{úsečka } AB \text{ v } \mathbb{R}^3 :$$

$$A = [2, -1, 1] \ , \ B = [1, 3, 3]$$

$$f(x,y,z) = x + y^2 - z \quad ;$$

$$\rightsquigarrow \text{parametrizácia } \varphi : \ \varphi(t) = A + t\cdot\overrightarrow{AB} = (2 + t\cdot(1-2) \ , -1 + t(3+1) \ , 1 + t(3-1))$$

$$= (\underbrace{2-t}_{x} \ , \ \underbrace{-1+4t}_{y} \ , \ \underbrace{1+2t}_{z}) \ , \ t \in [0,1]$$

$$\varphi'(t) = (-1, 4, 2) \rightsquigarrow \|\varphi'(t)\| = \sqrt{1+16+4} = \sqrt{21}$$

$$I = \int_0^1 \left(2-t + (4t-1)^2 - 1 - 2t\right)\sqrt{21}\, dt$$

$$= \int_0^1 \left(2-t + 16t^2 - 8t + 1 - 1 - 2t\right)\sqrt{21}\, dt = \int_0^1 \left(16t^2 - 11t + 2\right)\sqrt{21}\, dt$$

$$= \left[\frac{16}{3}t^3 - \frac{11}{2}t^2 + 2t\right]\sqrt{21} = \left(\frac{16}{3} - \frac{11}{2} + 2\right)\sqrt{21} = \boxed{\frac{11\sqrt{21}}{6}}$$

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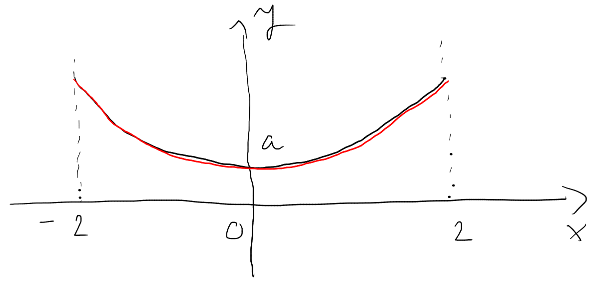
## APLIKÁCIE KI I. DRUHU

### 6. DĹŽKA KRIVKY $\rightsquigarrow$ $l = \int_\varphi ds$

určme dĺžku reťazovky $\rightsquigarrow$ graf $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) = a \cosh \frac{x}{a}$, $x \in [-2, 2]$, $a > 0$




$\rightsquigarrow$ parametrizácia $\varphi$: $x = t$, $y = \frac{a}{2}\left(e^{\frac{t}{a}} + e^{-\frac{t}{a}}\right)$, $t \in [-2, 2]$

$$\varphi'(t) = \left(1, \frac{a}{2}\left(\frac{1}{a}e^{\frac{t}{a}} - \frac{1}{a}e^{-\frac{t}{a}}\right)\right) = \left(1, \frac{e^{\frac{t}{a}} - e^{-\frac{t}{a}}}{2}\right)$$

$$\|\varphi'(t)\|^2 = 1 + \left(\frac{e^{\frac{t}{a}} - e^{-\frac{t}{a}}}{2}\right)^2 = 1 + \frac{e^{\frac{2t}{a}} + e^{\frac{-2t}{a}} - 2}{4} = \frac{e^{\frac{2t}{a}} + e^{-\frac{2t}{a}} + 2}{4}$$

$$= \frac{\left(e^{\frac{t}{a}} + e^{-\frac{t}{a}}\right)^2}{4} \Rightarrow \|\varphi'(t)\| = \frac{e^{\frac{t}{a}} + e^{-\frac{t}{a}}}{2}, \quad t \in [-2, 2]$$

$$\ell = \int_{-2}^{2} \frac{e^{\frac{t}{a}} + e^{-\frac{t}{a}}}{2}\,dt = \frac{1}{2}\left[a e^{\frac{t}{a}} - a e^{-\frac{t}{a}}\right]_{-2}^{2} = a\left[\sinh\frac{t}{a}\right]_{-2}^{2}$$

$$= \boxed{2a \sinh\frac{2}{a}}$$

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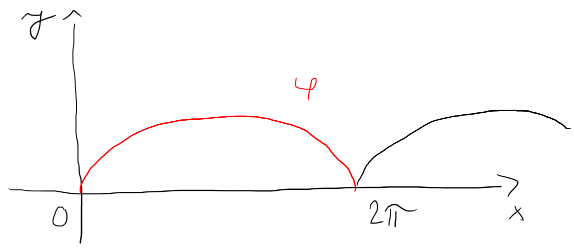
7.

délka jednoho oblouku cykloidy $\varphi$:

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t), \quad t \in [0, 2\pi]$$




$$x'(t) = a(1 - \cos t), \quad y'(t) = a \sin t$$

$$\|\varphi'(t)\| = \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t} = \sqrt{2a^2 - 2a^2 \cos t} = a \cdot \sqrt{2}\sqrt{1-\cos t}$$

$$= a\sqrt{2}\sqrt{2\sin^2\frac{t}{2}} = 2a\left|\sin\frac{t}{2}\right| = 2a \sin\frac{t}{2}, \quad t \in [0, 2\pi]$$

$$\ell = \int_0^{2\pi} 2a \sin\frac{t}{2}\,dt = \left[-4a\cos\frac{t}{2}\right]_0^{2\pi} = \boxed{8a}$$

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8.

délka jednoho závitu šroubovice $\varphi$: $x = a\cos t$, $y = a \sin t$, $z = bt$

$a, b > 0$

$\leadsto t \in [0, 2\pi]$; $\varphi'(t) = (-a\sin t, a\cos t, b)$, $\|\varphi'(t)\| = \sqrt{a^2 + b^2}$

$$\ell = \int_0^{2\pi} \sqrt{a^2+b^2}\,dt = \sqrt{a^2+b^2}\,\left[t\right]_0^{2\pi} = \boxed{2\pi\sqrt{a^2+b^2}}$$

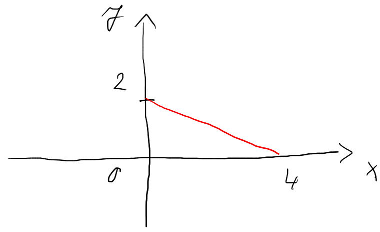
## 9. OBSAH VALCOVEJ PLOCHY POD GRAFOM F-CIE NAD KRIVKOU

$$S = \int_{\varphi} f(x,y)\,ds$$

obsah valcovej plochy: $x + 2y = 4$, $z = 4 - x - y$, $x, y \geq 0$

⇝ f-cia $f(x,y) = 4 - x - y$ vytína valcovú plochu $x + 2y = 4$

→ krivka $\varphi$ ⇝ prienik $x + 2y = 4$ s rovinou $xy$




⇒ úsečka: $x = 4 - 2t$, $y = t$, $t \in [0,2]$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$$

$$S = \int_0^2 \left(4 - [4 - 2t] - t\right)\sqrt{5}\,dt = \int_0^2 t\sqrt{5}\,dt = \sqrt{5}\left[\frac{t^2}{2}\right]_0^2 = \boxed{2\sqrt{5}}$$

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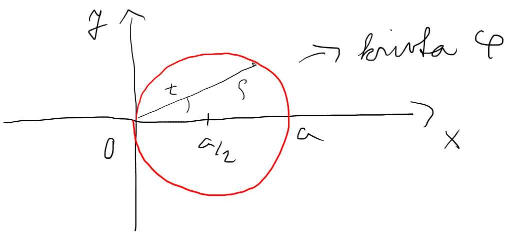
## 10.

obsah valcovej plochy: $x^2 + y^2 = ax$, $z = a + \frac{x^2}{a}$, $a \geq 0$

⇝ valcová plocha: $x^2 + y^2 = ax$ ⇝ $\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$




parametrizácia:

$$\left.\begin{aligned} x &= \rho\cos t = a\cos^2 t \\ y &= \rho\sin t = a\cos t\sin t \end{aligned}\right\} t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{(-2a\cos t\sin t)^2 + (a\cos 2t)^2} = a$$

⇝ f-cia $f(x,y) = z = a + \frac{x^3}{a}$

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(a + \frac{a^2 \cos^4 t}{a}\right) a \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(a^2 + a^2 \cos^4 t\right) dt =$$

$$= a^2 \left[ t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t \, dt = \pi a^2 + a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t \, dt \underset{=}{\overset{\text{POL. UHOL}}{}}$$

$$= \pi a^2 + a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1 + \cos 2t}{2}\right)^2 dt = \pi a^2 + a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{1}{4} + \frac{1}{2} \cos 2t + \frac{1}{4} \cos^2 2t \right) dt$$

$$= \pi a^2 + a^2 \left[ \frac{t}{4} + \frac{1}{4} \sin 2t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{a^2}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 2t \, dt$$

$$= \pi a^2 + \frac{\pi a^2}{4} + \frac{a^2}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 4t}{2} dt = \frac{5}{4} \pi a^2 + \frac{a^2}{4} \left[ \frac{t}{2} + \frac{1}{8} \sin 4t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{5 \pi a^2}{4} + \frac{a^2}{4} \cdot \frac{\pi}{2} = \boxed{\frac{11}{8} \pi a^2}$$

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