# KRIVKOVÝ INTEGRÁL II. DRUHU

$I = \int_\varphi \bar{f} \cdot d\bar{r}$ ; $\bar{f}$ je vektorová f-cia $n$ premenných definovaná na orientovanej krivke $\varphi : [a,b] \to \mathbb{R}^n$

⇝ pomocou parametrizácie $\varphi(t)$ prevedieme na určitý integrál

$$I = \pm \int_\varphi \bar{f}(\varphi(t)) \cdot \varphi'(t)\, dt$$

$+$ ⇝ parametrizácia je súhlasná s danou orientáciou

$-$ ⇝ ——||—— je nesúhlasná s danou orientáciou

⇝ klasický zápis:

$f = (f_1, \dots, f_n)$ , $d\bar{r} = (dx_1, \dots, dx_n)$

$$I = \int_\varphi f_1 dx_1 + f_2 dx_2 + \dots + f_n dx_n$$

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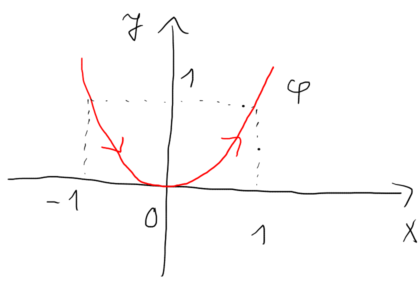
11.

$$I = \int_\varphi (x^2 - 2xy)\,dx + (y^2 - 2xy)\,dy \quad ; \quad \varphi : y = x^2 \; ; \; x \in [-1, 1]$$

orientovaná v smere rastu $x$




⇝ parametrizácia: $x = t$ , $y = t^2$ , $t \in [-1, 1]$

$\Rightarrow$ súhlasná s orientáciou

$\varphi(t) = (t, t^2)$ , $\varphi'(t) = (1, 2t)$

⇝ $dx = dt$ , $dy = 2t\,dt$

$$I = +\int_{-1}^{1} \Big( (t^2 - 2t^3)\,dt + (t^4 - 2t^3)\cdot 2t\,dt \Big) =$$

$$= \int_{-1}^{1} (t^2 - 2t^3 + 2t^5 - 4t^4)\,dt = \left[ \frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4}{5}t^5 \right]_{-1}^{1}$$

$$= \frac{2}{3} - \frac{8}{5} = \boxed{\frac{-14}{15}}$$

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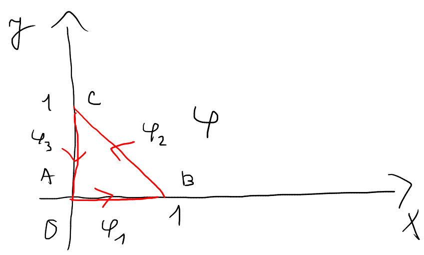
(12.)

$$I = \int_{\varphi} (x^2 + y^2)\,dx + (x^2 - y^2)\,dy \quad , \quad \varphi : \text{kladne orientovaný obvod}$$

$\Delta ABC$ , $A = [0,0]$ , $B = [1,0]$

$C = [0,1]$




$$\varphi = \varphi_1 \oplus \varphi_2 \oplus \varphi_3$$

$$I = I_1 + I_2 + I_3$$

a) $I_1 = \int_{\varphi_1} (x^2 + y^2)\,dx + (x^2 - y^2)\,dy$ ; $\varphi_1 : A \longrightarrow B$

$x = t$ , $y = 0$ , $t \in [0,1]$ $\rightsquigarrow$ parametrizácia súhlasná s orient.

$dx = dt$ , $dy = 0$

$$I_1 = \int_0^1 t^2\,dt = \left[ \frac{t^3}{3} \right]_0^1 = \frac{1}{3}$$

b) $I_2 = \int_{\varphi_2} (x^2 + y^2)\,dx + (x^2 - y^2)\,dy$ , $\varphi_2 : B \longrightarrow C$

$x = t$ , $y = 1 - t$ , $t \in [0,1]$ $\rightsquigarrow$ paramet. nesúhlasná s orientáciou

$dx = dt \quad , \quad dy = -dt$

$$I_2 = -\int_0^1 \left[ (t^2 + (1-t)^2)\,dt - (t^2 - (1-t)^2)\,dt \right]$$

$$= -\int_0^1 \left( 2t^2 - 2t + 1 - 2t + 1 \right) dt = -\int_0^1 \left( 2t^2 - 4t + 2 \right) dt$$

$$= -\left[ \tfrac{2}{3}t^3 - 2t^2 + 2t \right]_0^1 = -\frac{2}{3}$$

c) $I_3 = \int_{\varphi_3} (x^2+y^2)\,dx + (x^2 - y^2)\,dy \quad ; \quad \varphi_3 : C \to A$

$x = 0 \quad , \quad y = t \quad , \quad t \in [0,1] \rightsquigarrow$ param. nesúhlasná s orientáciou

$dx = 0 \quad , \quad dy = dt$

$$I_3 = -\int_0^1 (-t^2)\,dt = \int_0^1 t^2\,dt = \left[ \frac{t^3}{3} \right]_0^1 = \frac{1}{3}$$

$$I = I_1 + I_2 + I_3 = \frac{1}{3} - \frac{2}{3} + \frac{1}{3} = \boxed{0}$$

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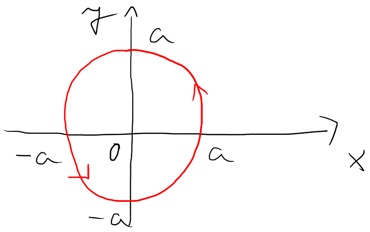
13.

$$I = \oint_{\varphi} \frac{x+y}{x^2+y^2}\,dx - \frac{(x-y)}{x^2+y^2}\,dy \quad , \quad \varphi : x^2+y^2 = a^2 \ , \ \circlearrowleft \quad a > 0$$

cirkulácia → integrál II. druhu po uzavretej krivke




$x = a\cos t$ , $y = a\sin t$ , $t \in [0, 2\pi]$

$\rightsquigarrow$ parametrizácia súhlasná s orient.

$dx = -a \sin t \, dt$ , $dy = a\cos t \, dt$

$x^2 + y^2 = a^2$

$$I = \int_0^{2\pi} \left[ \frac{a(\cos t + \sin t)}{a^2}(-a\sin t)dt - \frac{a(\cos t - \sin t)}{a^2}(a\cos t)\, dt \right]$$

$$= \int_0^{2\pi} \left( -\cos t \sin t - \sin^2 t - \cos^2 t + \sin t \cos t \right) dt = \int_0^{2\pi} (-1) dt$$

$$= \left[ -t \right]_0^{2\pi} = \boxed{-2\pi}$$

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14.

$$I = \int_\varphi y\,dx + z\,dy + x\,dz \quad , \quad \varphi : \text{ skrutkovica}$$

$x = a\cos t$ , $y = a \sin t$ , $z = bt$

$t \in [0, 2\pi]$ , $a, b > 0$

orientovaná súhlasne s param.

$dx = -a\sin t \, dt$ , $dy = a\cos t \, dt$ , $dz = b\,dt$

$$I = \int_0^{2\pi} \left[ a\sin t\,(-a\sin t)dt + bt(a\cos t)dt + a\cos t\,(b)\,dt \right] =$$

$$= \int_0^{2\pi} \left( -a^2 \sin^2 t + ab\, t\cos t + ab\cos t \right) dt =$$

$$= \int_0^{2\pi} \left( -\frac{a^2}{2} + \frac{a^2}{2}\cos 2t + ab(t+1)\cos t \right) dt$$

$$= \left[-\frac{a^2}{2}t + \frac{a^2}{4}\sin 2t\right]_0^{2\pi} + ab\int_0^{2\pi}(t+1)\cos t\,dt$$

$$= -\pi a^2 + ab\int_0^{2\pi}(t+1)\cos t\,dt = \left|\begin{array}{ll} u = t+1 , & v' = \cos t \\ u' = 1 , & v = \sin t \end{array}\right|$$

$$= -\pi a^2 + ab\left[(t+1)\sin t\right]_0^{2\pi} - ab\int_0^{2\pi}\sin t\,dt =$$

$$= -\pi a^2 - ab\left[-\cos t\right]_0^{2\pi} = \boxed{-\pi a^2}$$

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15. $I = \int_\varphi \frac{dz}{x^2+y^2+z^2}$ ; $\varphi$ : priesek plôch $x^2+y^2=4$, $x+z=2$, orientovaná z pohľadu kladnej časti osi $z$ proti smeru hod. ručičiek od $[2,0,0]$ do $[-2,0,4]$

$\rightsquigarrow$ parametrizácia $\varphi$ : $x = 2\cos t$ , $y = 2\sin t$ , $z = 2 - x = 2 - 2\cos t$

pre $[2,0,0]$ : $t = 0$

pre $[-2,0,4]$ : $t = \pi$ $\Rightarrow$ paramet. súhlasná s orientáciou

(v rovine $xy$ sa otáčame kladne, t.j., v smere rastu $t$)

$dx = -2\sin t\,dt$ , $dy = 2\cos t\,dt$ , $dz = 2\sin t\,dt$

$$I = \int_0^{\pi}\frac{2\sin t\,dt}{4+4(1-\cos t)^2} = \frac{1}{2}\int_0^{\pi}\frac{\sin t\,dt}{1+(1-\cos t)^2} =$$

$$= \left| \begin{array}{ll} u = 1 - \cos t, & du = \sin t \, dt \\ 0 \rightsquigarrow 0, & \pi \rightsquigarrow 2 \end{array} \right| = \frac{1}{2} \int_0^2 \frac{du}{1+u^2}$$

$$= \frac{1}{2} \left[ \operatorname{arctg} u \right]_0^2 = \boxed{\frac{\operatorname{arctg} 2}{2}}$$

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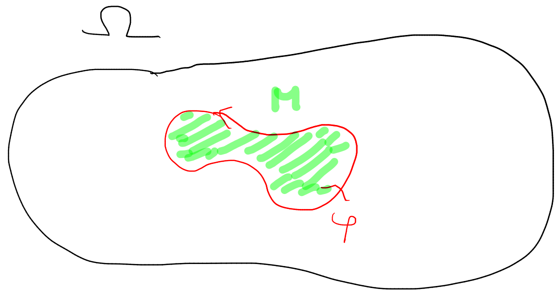
## GREENOVA INTEGRÁLNA VETA




$P(x,y)$, $Q(x,y)$, $P'_y(x,y)$, $Q'_x(x,y)$ sú spojité na oblasti $\Omega$

$\overline{M} := \operatorname{Int} \varphi \cup \langle \varphi \rangle$

$$\oint_{\varphi} P(x,y)\,dx + Q(x,y)\,dy = \iint_{\overline{M}} \left( Q'_x(x,y) - P'_y(x,y) \right) dx\,dy$$

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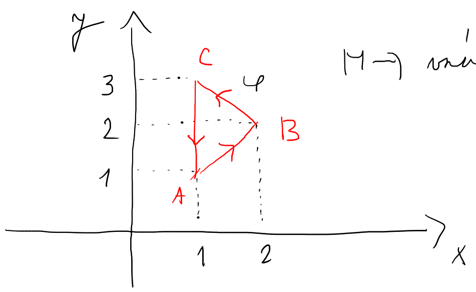
16. $I = \oint_{\varphi} 2(x^2+y^2)\,dx + (x+y)^2\,dy$ ; $\varphi$: kladne orientovaný obvod $\triangle ABC$; $A = [1,1]$, $B = [2,2]$, $C = [1,3]$




$M \to$ vnútro $\triangle ABC$

$P(x,y) = 2(x^2+y^2)$, $Q(x,y) = (x+y)^2$

$P'_y = 4y$, $Q'_x = 2(x+y)$

$\rightsquigarrow$ sú spojité na $\mathbb{R}^2$

$\Rightarrow$ na výpočet $I$ môžeme aplikovať Greenovu vetu:

$$I = \iint_{\overline{M}} [2(x+y) - 4y]\,dx\,dy = \iint_{\overline{M}} 2(x-y)\,dx\,dy$$

$\overline{M}$ je elementárna oblasť vzhľadom na $x$:

$$1 \le x \le 2 \quad ; \quad x \le y \le 4-x$$

$$I = \int_1^2 \left[ \int_x^{4-x} 2(x-y)\,dy \right] dx = \int_1^2 \left[ 2xy - y^2 \right]_x^{4-x} dx =$$

$$= \int_1^2 \left( 2x(4-x) - (4-x)^2 - 2x^2 + x^2 \right) dx = \int_1^2 \left( -4x^2 + 16x - 16 \right) dx$$

$$= -4 \int_1^2 (x^2 - 4x + 4)\,dx = -4 \int_1^2 (x-2)^2\,dx = -4 \left[ \frac{(x-2)^3}{3} \right]_1^2 =$$

$$= \boxed{-\frac{4}{3}}$$

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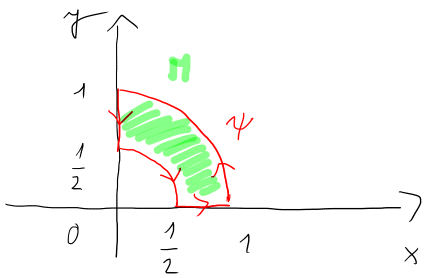
17.

$$I = \oint_{\psi} \sqrt{x^2+y^2}\,dx + y\left(xy + \ln\left(\sqrt{x^2+y^2} + x\right)\right) dy \quad ;$$

$\psi$ je kladne orientovaná hranica oblasti $M$:

$$\frac{1}{4} \le x^2 + y^2 \le 1 \quad , \quad x, y \ge 0$$




$$P(x,y) = \sqrt{x^2+y^2} \quad , \quad P'_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

$$Q(x,y) = y\left(xy + \ln\left(\sqrt{x^2+y^2} + x\right)\right) \quad ,$$

$$Q'_x(x,y) = y^2 + \frac{y}{\sqrt{x^2+y^2}}$$

$\Rightarrow$ sú spojité v $\mathbb{R}_0^{2+} \setminus \{[0,0]\}$

⇝ aplikujeme Greenovu vetu:

$$I = \iint_{\overline{M}} \left( y^2 + \frac{y}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}} \right) dx\,dy = \iint_{\overline{M}} y^2\,dx\,dy$$

⇝ polárne súradnice: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $J = \rho$

$$\overline{M}: \quad 0 \le \varphi \le \frac{\pi}{2}, \quad \frac{1}{2} \le \rho \le 1$$

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$$I = \int_0^{\pi/2} \left[ \int_{1/2}^{1} \rho^2 \sin^2\varphi \cdot \rho\,d\rho \right] d\varphi = \left( \int_0^{\pi/2} \sin^2\varphi\,d\varphi \right) \left( \int_{1/2}^{1} \rho^3\,d\rho \right)$$

$$= \left( \int_0^{\pi/2} \frac{1-\cos 2\varphi}{2}\,d\varphi \right) \left[ \frac{\rho^4}{4} \right]_{1/2}^{1} = \left[ \frac{\varphi}{2} - \frac{\sin 2\varphi}{4} \right]_0^{\frac{\pi}{2}} \cdot \left( \frac{1}{4} - \frac{1}{64} \right)$$

$$= \frac{\pi}{4} \cdot \frac{15}{64} = \boxed{\frac{15\pi}{256}}$$

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## APLIKÁCIE KRIVKOVÉHO INTEGRÁLU II. DRUHU

18. obsah vnútra Jordanovej krivky $\varphi$ kladne orientovanej:

$$\boxed{S = \frac{1}{2} \oint_{\varphi} x\,dy - y\,dx}$$

ODVODENIE ⇝ pomocou GREENOVEJ VETY

$P(x,y) := -y \quad , \quad Q(x,y) := x$

$P'_y = -1 \quad , \quad Q'_x = 1$

$\Big\}$ spojité na $\mathbb{R}^2$

$\rightsquigarrow$ platí:

$$\frac{1}{2} \oint_{\varphi} x\,dy - y\,dx = \frac{1}{2} \iint_{\operatorname{Int}\varphi \cup \langle\varphi\rangle} (1-(-1))\,dx\,dy = \iint_{\operatorname{Int}\varphi \cup \langle\varphi\rangle} dx\,dy = S$$

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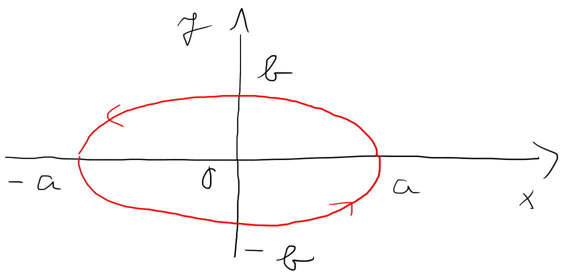
19.

obsah vnútra elipsy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , $a, b > 0$




elipsa $\varphi$ je Jordanova krivka

$$\Rightarrow S = \frac{1}{2} \oint_{\varphi} x\,dy - y\,dx$$

$\rightsquigarrow$ parametrizácia: $x = a\cos t$ , $y = b\sin t$ , $t \in [0, 2\pi]$

$\Rightarrow$ súhlasná s kladnou orientáciou

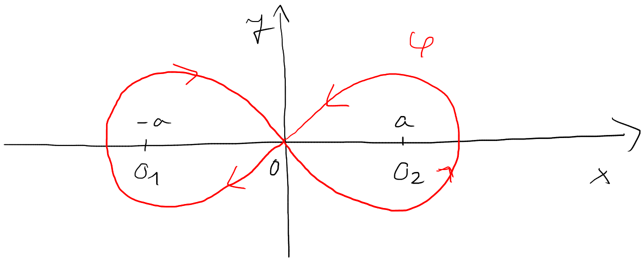
$dx = -a\sin t\,dt$ , $dy = b\cos t\,dt$

$$S = \frac{1}{2} \int_0^{2\pi} \big(a\cos t\,(b\cos t)\,dt - b\sin t\,(-a\sin t)\,dt\big) = \frac{1}{2}\int_0^{2\pi} ab\,dt$$

$$= \frac{1}{2} ab \cdot [t]_0^{2\pi} = \boxed{\pi ab}$$

(20.)

**Bernoulliho lemniskáta** $\rightsquigarrow$ množina bodov $A$ v $\mathbb{R}^2$, pre ktoré je súčin ich vzdialeností od daných dvoch bodov $O_1, O_2$; $|O_1O_2| = 2a$, rovný $a^2$




$\rightsquigarrow$ parametrické vyjadrenie: $x = \dfrac{a\sqrt{2}\cos t}{1+\sin^2 t}$, $y = \dfrac{a\sqrt{2}\cos t \sin t}{1+\sin^2 t}$

$$t \in [0, 2\pi]$$

$\rightsquigarrow$ určíme obsah jedného „oka" lemniskáty pre $x \geq 0$:

$$x = \frac{a\sqrt{2}\cos t}{1+\sin^2 t}, \quad y = \frac{a\sqrt{2}\cos t \sin t}{1+\sin^2 t}; \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$dx = a\sqrt{2}\cdot\left(\frac{\sin^3 t - 3\sin t}{(1+\sin^2 t)^2}\right)dt, \quad dy = a\sqrt{2}\left(\frac{1-3\sin^2 t}{(1+\sin^2 t)^2}\right)dt$$

$$I = \frac{1}{2}\int_{\varphi} x\,dy - y\,dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[2a^2\cdot\frac{\cos t\,(1-3\sin^2 t) - \sin t\cos t\,(\sin^3 t - 3\sin t)}{(1+\sin^2 t)^3}\right]dt$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos t\,(1-\sin^4 t)}{(1+\sin^2 t)^3}\,dt = a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(1+\sin^2 t)(1-\sin^2 t)}{(1+\sin^2 t)^3}\cos t\,dt =$$

$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\sin^2 t}{(1+\sin^2 t)^2} \cos t \, dt = \left| \begin{array}{ll} u = \sin t, & du = \cos t \, dt \\ -\pi/2 \rightsquigarrow -1, & \pi/2 \rightsquigarrow 1 \end{array} \right|$$

$$= a^2 \int_{-1}^{1} \frac{1-u^2}{(1+u^2)^2} \, du = a^2 \cdot \left[ \frac{u}{1+u^2} \right]_{-1}^{1} = \boxed{a^2}$$

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(21.) PRÁCA SILOVÉHO POĽA

$\rightsquigarrow$ silové pole $\bar{f}(x,y,z)$ pôsobiace po orientovanej krivke $\varphi$ vykoná prácu: $\underline{W = \int_{\varphi} \bar{f}(x,y,z) \cdot d\bar{r}}$

$f(x,y,z) = \left(0, 0, \frac{z}{x^2+y^2+z^2}\right)$ ; $\varphi$: prienik valcových plôch

$x^2+y^2 = 4$, $x^2+(z-2)^2 = 4$, $z \leq 2$

krivka $\varphi$ je orientovaná vzhľadom na kladný smer osi $z$ proti smeru hodinových ručičiek ; prechod od bodu $[2,0,2]$ do bodu $[0,2,0]$ v smere orientácie $\varphi$

$\rightsquigarrow$ parametrizácia $\varphi(t)$ : $x = 2\cos t$, $y = 2\sin t$, $2-z = 2|\sin t|$

$[2,0,2] \rightsquigarrow$ hodnota parametra $t=0$

$[0,2,0] \rightsquigarrow$ hodnota parametra $t=\frac{\pi}{2}$

$\rightsquigarrow t \in [0, \frac{\pi}{2}] \Rightarrow z = 2(1-\sin t)$

$\Rightarrow$ parametrizácia súhlasná s orientáciou

$dx = -2\sin t \, dt$, $dy = 2\cos t \, dt$, $dz = -2\cos t \, dt$

$$W = \int_0^{\frac{\pi}{2}} \frac{2(1-\sin t)}{4+4(1-\sin t)^2}\cdot(-2\cos t)\,dt = \int_0^{\frac{\pi}{2}} \frac{1-\sin t}{1+(1-\sin t)^2}(-\cos t)\,dt$$

$$= \left| \begin{array}{l} u = 1-\sin t\,,\ du = -\cos t\,dt \\ 0 \rightsquigarrow 1\,,\ \frac{\pi}{2} \rightsquigarrow 0 \end{array} \right| = \int_1^0 \frac{u}{1+u^2}\,du =$$

$$= \left[\frac{1}{2}\ln(1+u^2)\right]_1^0 = \boxed{-\frac{1}{2}\ln 2}$$

práca pri pohybe sa
koná na úkor poľa

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(22.) rovnaké zadanie ako v (21.), ale teraz v smere orientácie φ od bodu $\underbrace{[0,2,0]}_{t=\frac{\pi}{2}}$ do bodu $\underbrace{[-2,0,2]}_{t=\pi}$ $\rightsquigarrow t \in [\frac{\pi}{2}, \pi] \rightsquigarrow$

paramet. je opäť súhlasná s orientáciou a $r = 2(1-\sin t)$

$$W = \int_{\frac{\pi}{2}}^{\pi} \frac{1-\sin t}{1+(1-\sin t)^2}(-\cos t)\,dt = \left| \begin{array}{l} u = 1-\sin t\,,\ du = -\cos t\,dt \\ \frac{\pi}{2} \rightsquigarrow 0\ ;\ \pi \rightsquigarrow 1 \end{array} \right|$$

$$= \int_0^1 \frac{u}{1+u^2}\,du = \left[\frac{1}{2}\ln(1+u^2)\right]_0^1 = \boxed{\frac{1}{2}\ln 2}$$

$\rightsquigarrow$ prácu pri pohybe koná pole

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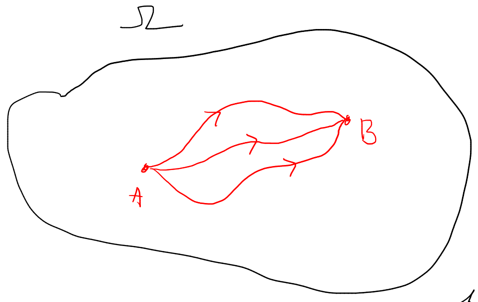
NEZÁVISLOSŤ KRIVKOVÉ INTEGRÁLU
NA INTEGRAČNEJ CESTE




$\bar f(x,y,z)$ je spojitá vektorová f-cia (pole)

$\Rightarrow$ nezávislosť $\int_\varphi \bar f \cdot d\bar r$ znamená, že

pole $\bar f$ je POTENCIÁLOVÉ v $\Omega$, t.j.,

$\bar f(x,y,z) = \operatorname{grad} V(x,y,z)$,

$V$ je skalárna f-cia, tzv. POTENCIÁL poľa $\bar f$

$\rightarrow \int_\varphi \bar f \cdot d\bar r = V(B) - V(A)$ pre $\forall$ krivku $\varphi$, ktorá spája $A, B$, orientovanú od $A$ do $B$

$\leadsto$ rovinné pole $f(x,y) = (P(x,y), Q(x,y))$ :

ak $f(x,y)$ je potenciálové v $\Omega \subseteq \mathbb{R}^2 \Rightarrow P'_y(x,y) = Q'_x(x,y)$ v $\Omega$

ak naviac oblasť $\Omega$ je jednoducho súvislá, potom :

ak $P'_y(x,y) = Q'_x(x,y)$ v $\Omega \Rightarrow f(x,y)$ je potenciálové v $\Omega$

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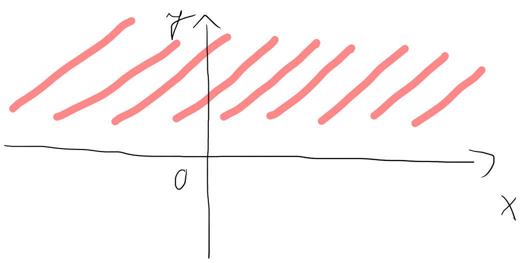
(23.)

$$f(x,y) = \left( -\frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}} \right); \quad \Omega = \{[x,y] \in \mathbb{R}^2 ; y > 0\}$$

Je pole $f$ potenciálové v $\Omega$ ?




$$P = -\frac{x}{\sqrt{x^2+y^2}}, \quad Q = \frac{-y}{\sqrt{x^2+y^2}}$$

$\leadsto$ $P, Q$ sú spojité na $\Omega$

$\leadsto$ $\Omega$ je jednoducho súvislá oblasť

$\Rightarrow$ stačí overiť: $P'_y = Q'_x$

$P'_y = \frac{xy}{(x^2+y^2)^{3/2}}$ , $Q'_x = \frac{yx}{(x^2+y^2)^{3/2}}$ $\leadsto$ platí

$\leadsto$ pole $f(x,y)$ je potenciálové v $\Omega$

potenciál $V$ (kmeňová f-cia pre dvojicu $P, Q$):

$$\underline{V'_x = P \ , \ V'_y = Q}$$

$$V(x,y) = \int P(x,y)\,dx = \int -\frac{x}{\sqrt{x^2+y^2}}\,dx = -\int \frac{x\,dx}{\sqrt{x^2+y^2}} =$$

$$= \left| u = x^2+y^2 \ ; \ du = 2x\,dx \right| = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + C(y)$$

$$= -\sqrt{x^2+y^2} + C(y)$$

$\leadsto$ $V'_y = -\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y + C'(y) = -\frac{y}{\sqrt{x^2+y^2}} + C'(y)$

$V'_y = Q = -\frac{y}{\sqrt{x^2+y^2}}$

$\Rightarrow$ f-cia $C(y)$ spĺňa: $C'(y) = 0 \Rightarrow C(y) \equiv K$

$\leadsto$ $\boxed{V(x,y) = -\sqrt{x^2+y^2} + K \ ; \ K \in \mathbb{R}}$

24.

$$f(x,y) = \left( \frac{-y}{x^2+y^2} , \frac{x}{x^2+y^2} \right) ; \quad \Omega = \mathbb{R}^2 \setminus \{[0,0]\}$$

Je pole $f$ potenciálové v oblasti $\Omega$ ?

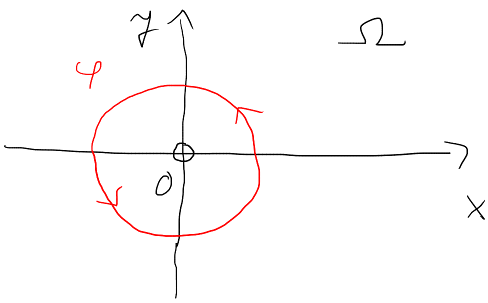
⇝ $P = \dfrac{-y}{x^2+y^2}$ , $Q = \dfrac{x}{x^2+y^2}$ sú spojité v $\Omega$

⇝ $\Omega$ nie je jednoducho súvislá oblasť

⇝ platí: $P'_y = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ ; $Q'_x = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$\Rightarrow$ platí rovnosť $P'_y = Q'_x$ , ale teraz to nemusí stačiť na to, aby $f$ bolo potenciálové pole

⇝ pole $f$ v skutočnosti nie je potenciálové




$\varphi: x^2+y^2 = 4$ , ↺

parametrizácia:

$x = 2\cos t$ , $y = 2\sin t$ , $t \in [0, 2\pi]$

⇝ súhlasná s orientáciou

$dx = -2\sin t \, dt$ , $dy = 2\cos t \, dt$

$$I = \oint_\varphi P(x,y)\,dx + Q(x,y)\,dy =$$

$$= \int_0^{2\pi} \left[ \frac{-2\sin t \cdot (-2\sin t)\,dt}{4} + \frac{2\cos t \cdot (2\cos t)\,dt}{4} \right] =$$

$$= \int_0^{2\pi} \left( \sin^2 t + \cos^2 t \right) dt = \int_0^{2\pi} dt = \left[ t \right]_0^{2\pi} = 2\pi \neq 0$$

POZOR !!! $\rightsquigarrow$ pri výpočte I sme nemohli použiť Greenovu vetu, pretože P, Q nie sú spojité všade vo vnútri kružnice $\varphi$

---

25. $$I = \int_{\varphi} (2y - 6xy^3)\, dx + (2x - 9x^2y^2)\, dy$$ nezávisí v $\mathbb{R}^2$

na integračnej ceste $\varphi$, pretože:

- $\mathbb{R}^2$ je jednoducho súvislá oblasť
- $P = 2y - 6xy^3$, $Q = 2x - 9x^2y^2$

  $P'_y = 2 - 18xy^2$, $Q'_x = 2 - 18xy^2 \Rightarrow P'_y = Q'_x$

$\rightsquigarrow$ kmeňová f-cia $V(x,y)$:

$$V'_x = P, \quad V'_y = Q$$

$$V(x,y) = \int Q(x,y)\, dy = \int (2x - 9x^2y^2)\, dy =$$

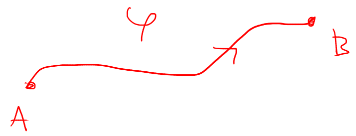
$$= 2xy - 3x^2y^3 + C(x)$$

$$V'_x = 2y - 6xy^3 + C'(x); \quad V'_x = 2y - 6xy^3$$

$$\Rightarrow C'(x) = 0 \rightsquigarrow C(x) \equiv K$$

$$\boxed{V(x,y) = 2xy - 3x^2y^3 + K, \quad K \in \mathbb{R}}$$

krivkový integrál $I$ po orientovanej krivke $\varphi$




má hodnotu : $\underline{I = V(B) - V(A)}$

→ napr. pre $A = [0,0]$, $B = [2,2]$

a) $\varphi$ je orientovaná úsečka $A \to B$, t.j.,

$$x = t, \quad y = t, \quad t \in [0,2], \quad dx = dt = dy$$

$$I = \int_0^2 \left(2t - 6t^4 + 2t - 9t^4\right) dt = \int_0^2 \left(4t - 15t^4\right) dt$$

$$= \left[2t^2 - 3t^5\right]_0^2 = \underline{-88} = V(B) - V(A)$$

b) $\varphi$ je parabola $2x = y^2$ orientovaná od $A$ do $B$

$$x = \frac{t^2}{2}, \quad y = t, \quad t \in [0,2], \quad dx = t\,dt, \quad dy = dt$$

$$I = \int_0^2 \left[\left(2t - 3t^5\right) t\,dt + \left(t^2 - \frac{9}{4}t^6\right) dt\right] = \int_0^2 \left(3t^2 - \frac{21}{4}t^6\right) dt$$

$$= \left[t^3 - \frac{3}{4} \cdot t^7\right]_0^2 = -88$$

c) $\varphi$ je kubická parabola $4y = x^3$ orientovaná od $A$ do $B$

$$x = t, \quad y = \frac{t^3}{4}, \quad t \in [0,2], \quad dx = dt, \quad dy = \frac{3}{4}t^2\,dt$$

$$I = \int_0^2 \left[ \left( \frac{t^3}{2} - \frac{3}{32} t^{10} \right) dt + \left( 2t - \frac{9}{16} t^8 \right) \cdot \frac{3}{4} t^2 \, dt \right] =$$

$$= \int_0^2 \left( 2t^3 - \frac{33}{64} t^{10} \right) dt = \left[ \frac{t^4}{2} - \frac{3}{64} t^{11} \right]_0^2 = -88$$

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(26.) $I = \int_{\varphi} (2x+3y)\,dx + (3x-4y)\,dy \rightsquigarrow$ závisí / nezávisí ?

- $\mathbb{R}^2$ je jednoducho súvislá oblasť
- $P = 2x+3y$ , $Q = 3x-4y$

  $P'_y = 3$ ; $Q'_x = 3$ $\rightsquigarrow$ $P'_y = Q'_x$

$\Rightarrow$ nezávisí na integračnej ceste $\varphi$

$\rightsquigarrow$ potenciál $V(x,y)$ :

$V'_x = P$ , $V'_y = Q$

$V = \int P\,dx = \int (2x+3y)\,dx = x^2 + 3yx + C(y)$

$V'_y = 3x + C'(y)$ , $V'_y = 3x - 4y$

$\rightsquigarrow$ $3x + C'(y) = 3x - 4y \Rightarrow C'(y) = -4y$

$C(y) = \int -4y\,dy = -2y^2 + K$

$\Rightarrow \boxed{V(x,y) = x^2 + 3xy - 2y^2 + K \ , \ K \in \mathbb{R}}$

---

27.

$$I = \int_{\varphi} \left(2xy^2 + 3x^2 + \frac{1}{x^2} + \frac{2x}{y^2}\right) dx + \left(2x^2y + 3y^2 + \frac{1}{y^2} - \frac{2x^2}{y^3}\right) dy$$

závisí / nezávisí v $\Omega := \{[x,y] \in \mathbb{R}^2 \; ; \; x,y > 0\}$

- $\Omega$ je jednoducho súvislá oblasť
- $P = 2xy^2 + 3x^2 + \frac{1}{x^2} + \frac{2x}{y^2}$ , $Q = 2x^2y + 3y^2 + \frac{1}{y^2} - \frac{2x^2}{y^3}$

$P'_y = 4xy - \frac{4x}{y^3}$ , $Q'_x = 4xy - \frac{4x}{y^3} \Rightarrow P'_y = Q'_x$

$\Rightarrow$ I nezávisí v $\Omega$ na integračnej ceste $\varphi$

$\rightsquigarrow$ potenciál $V(x,y)$:

$V'_x = P$ , $V'_y = Q$ ;

$$V(x,y) = \int P \, dx = \int \left(2xy^2 + 3x^2 + \frac{1}{x^2} + \frac{2x}{y^2}\right) dx =$$

$$= x^2y^2 + x^3 - \frac{1}{x} + \frac{x^2}{y^2} + C(y)$$

$V'_y = 2x^2y - \frac{2x^2}{y^3} + C'(y)$ , $V'_y = 2x^2y + 3y^2 + \frac{1}{y^2} - \frac{2x^2}{y^3}$

$$\Rightarrow 2x^2y - \frac{2x^2}{y^3} + C'(y) = 2x^2y + 3y^2 + \frac{1}{y^2} - \frac{2x^2}{y^3}$$

$\rightsquigarrow C'(y) = 3y^2 + \frac{1}{y^2} \rightsquigarrow C(y) = \int \left(3y^2 + \frac{1}{y^2}\right) dy \rightsquigarrow$

$C(y) = y^3 - \frac{1}{y} + K$ , $K \in \mathbb{R}$

$$\boxed{V(x,y) = x^2y^2 + x^3 - \frac{1}{x} + \frac{x^2}{y^2} + y^3 - \frac{1}{y} + K, \quad K \in \mathbb{R}}$$

---

28.

$$I = \int_{\varphi} (x^2 - 2yz)\,dx + (y^2 - 2xz)\,dy + (z^2 - 2xy)\,dz$$

<u>závisí / nezávisí v $\mathbb{R}^3$</u>

- $\mathbb{R}^3$ je jednoducho súvislá oblasť
- $P = x^2 - 2yz$, $Q = y^2 - 2xz$, $R = z^2 - 2xy$

$\rightsquigarrow$ nezávislosť je potom ekvivalentná s:

$$\boxed{P'_y = Q'_x, \quad P'_z = R'_x, \quad Q'_z = R'_y}$$

$P'_y = -2z$, $Q'_x = -2z$ $\rightsquigarrow$ $P'_y = Q'_x$ ✓

$P'_z = -2y$, $R'_x = -2y$ $\rightsquigarrow$ $P'_z = R'_x$ ✓

$Q'_z = -2x$, $R'_y = -2x$ $\rightsquigarrow$ $Q'_z = R'_y$ ✓

$\Rightarrow$ <u>$I$ nezávisí v $\mathbb{R}^3$ na integračnej ceste $\varphi$</u>

$\rightsquigarrow$ potenciál $V(x,y,z)$:

$V'_x = P$, $V'_y = Q$, $V'_z = R$

$$V(x,y,z) = \int P\,dx = \int (x^2 - 2yz)\,dx = \frac{x^3}{3} - 2xyz + C(y,z)$$

$$V'_y = -2xz + C'_y \quad , \quad V'_z = -2xy + C'_z$$

$$V'_y = y^2 - 2xz \quad , \quad V'_z = z^2 - 2xy$$

$\Rightarrow$ platí: $\quad -2xz + C'_y = y^2 - 2xz \rightsquigarrow C'_y = y^2$

$$-2xy + C'_z = z^2 - 2xy \rightsquigarrow C'_z = z^2$$

$$\rightsquigarrow C(y,z) = \int y^2\,dy = \frac{y^3}{3} + D(z)$$

$$C'_z = D'(z) \quad , \quad C'_z = z^2 \Rightarrow D'(z) = z^2 \rightsquigarrow D = \frac{z^3}{3} + K$$

$$K \in \mathbb{R}$$

$$\rightsquigarrow \underline{V(x,y,z)} = \frac{x^3}{3} - 2xyz + \frac{y^3}{3} + \frac{z^3}{3} + K$$

$$= \boxed{\frac{x^3 + y^3 + z^3}{3} - 2xyz + K \quad , K \in \mathbb{R}}$$

---

29.

$$I = \int_{\varphi} \left(3x^2 + \frac{y}{(x+z)^2}\right) dx - \frac{dy}{x+z} + \frac{y}{(x+z)^2}\,dz$$

závisí / nezávisí v oblasti $\Omega = \{[x,y,z] \in \mathbb{R}^3 , \ x+z < 0\}$

• $\Omega$ je jednoducho súvislá oblasť

• $P = 3x^2 + \frac{y}{(x+z)^2}$ , $Q = -\frac{1}{x+z}$ , $R = \frac{y}{(x+z)^2}$

$P'_y = \frac{1}{(x+z)^2}$ , $Q'_x = \frac{1}{(x+z)^2}$ ✓

$P'_z = -\frac{2y}{(x+z)^3}$ , $R'_x = -\frac{2y}{(x+z)^3}$ ✓

$Q'_z = \frac{1}{(x+z)^2}$ , $R'_y = \frac{1}{(x+z)^2}$ ✓

$\rightsquigarrow$ I nezávislosť na integračnej ceste v $\Omega$

$\rightsquigarrow$ potenciál $V(x,y,z)$:

$V'_x = P$ , $V'_y = Q$ , $V'_z = R$

$$V = \int R \, dz = \int \frac{y}{(x+z)^2} dz = -\frac{y}{x+z} + C(x,y)$$

$V'_x = \frac{y}{(x+z)^2} + C'_x$ , $V'_x = 3x^2 + \frac{y}{(x+z)^2} \Rightarrow C'_x = 3x^2$

$V'_y = -\frac{1}{x+z} + C'_y$ , $V'_y = -\frac{1}{x+z} \Rightarrow C'_y = 0$

$\rightsquigarrow$ $C(x,y) = x^3 + K$ , $K \in \mathbb{R}$

$$\boxed{V(x,y,z) = -\frac{y}{x+z} + x^3 + K \; , \quad K \in \mathbb{R}}$$