PLOŠNÉ INTEGRÁLY I. DRUHU

$I = \int_\sigma f\,dS$; f je funkcia troch premenných definovaná na ploche σ ⟶ je daná nejakou parametrizáciou $\vec{r} : M \to \mathbb{R}^3$, M je oblasť v $\mathbb{R}^2$, t.j.

$$\vec{r} = \vec{r}(u,v) \in \mathbb{R}^3 \; ; \; [u,v] \in M$$

⟶ plošný integrál prevedieme na dvojný integrál cez oblasť M:

$$I = \iint_M f(\vec{r}(u,v)) \cdot \| \vec{r}\,'_u(u,v) \times \vec{r}_v(u,v) \| \, du\, dv ,$$

$\| \cdot \|$ je euklidovská norma v $\mathbb{R}^3$ a $\pm(\vec{r}_u \times \vec{r}_v)$ je normálový vektor plochy σ

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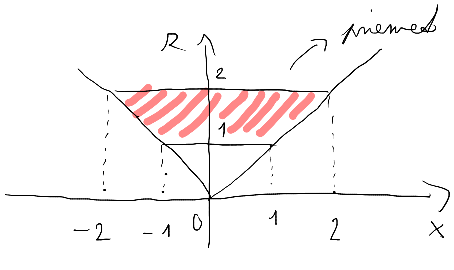
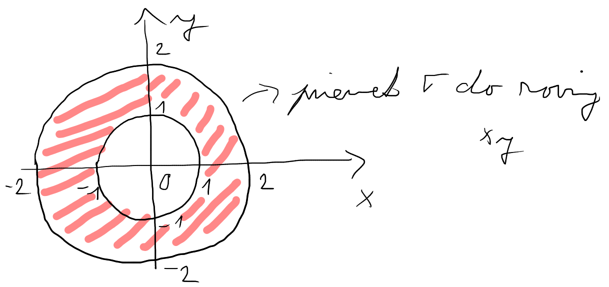
1. $I = \int_\sigma z\, dS$; σ : časť kužeľovej plochy $z = \sqrt{x^2+y^2}$; $1 \le z \le 2$





⟶ parametrizácia $\vec{r}(u,v)$:

$$x = u \, , \; y = v \, , \; z = \sqrt{u^2+v^2} \; ; \; M : 1 \le u^2 + v^2 \le 4$$

⟶ $\vec{r} = \left(u, v, \sqrt{u^2+v^2} \right)$

⟶ $\vec{r}\,'_u = \left(1, 0, \frac{u}{\sqrt{u^2+v^2}} \right)$, $\vec{r}\,'_v = \left(0, 1, \frac{v}{\sqrt{u^2+v^2}} \right)$

$\rightsquigarrow$ normálový vektor $\bar{n}(u,v)$ plochy :

$$\bar{n}(u,v) = \bar{r}'_u(u,v) \times \bar{r}'_v(u,v)$$

$$\rightsquigarrow \bar{r}'_u(u,v) \times \bar{r}'_v(u,v) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix}$$

$$= \frac{-u}{\sqrt{u^2+v^2}}\bar{i} - \frac{v}{\sqrt{u^2+v^2}}\bar{j} + \bar{k}$$

$$\Rightarrow \bar{n}(u,v) = \left(\frac{-u}{\sqrt{u^2+v^2}}, \frac{-v}{\sqrt{u^2+v^2}}, 1 \right)$$

$$\Rightarrow \|\bar{n}(u,v)\| = \sqrt{\frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2} + 1} = \sqrt{2}$$

$$\rightsquigarrow I = \iint_M \sqrt{u^2+v^2} \cdot \sqrt{2}\, du\, dv \quad ; \quad M: \; 1 \leq u^2+v^2 \leq 4$$

polárne súradnice : $u = \rho\cos\varphi$, $v = \rho\sin\varphi$, $J = \rho$

$$\varphi \in [0, 2\pi] \quad ; \quad 1 \leq \rho \leq 2$$

$$I = \int_0^{2\pi} \left[\int_1^2 \rho^2 \sqrt{2}\, d\rho \right] d\varphi = \sqrt{2} \int_0^{2\pi} d\varphi \cdot \int_1^2 \rho^2 d\rho = 2\pi\sqrt{2} \left[\frac{\rho^3}{3} \right]_1^2$$

$$= \boxed{\frac{14\sqrt{2}\pi}{3}}$$

(2.)

$$I = \int_{\sigma} x^2 y^2 \, dS \quad , \quad \sigma: \; x^2 + y^2 + z^2 = R^2 \; ; \; z \geq 0 \; , \; R > 0$$

$f(x,y,z) = x^2 y^2$; parametrizácia plochy σ :

$$x = R\cos u \sin v \; , \; y = R \sin u \sin v \; , \; z = R\cos v \; ,$$

$$u \in [0, 2\pi] \; , \; v \in [0, \pi/2]$$

$$\Rightarrow \; \bar{r}(u,v) = (R\cos u \sin v \; , \; R\sin u \sin v \; , \; R\cos v) \; ; \; [u,v] \in M \, ,$$

kde $M: \; 0 \leq u \leq 2\pi \; , \; 0 \leq v \leq \dfrac{\pi}{2}$

$$\rightsquigarrow \; \bar{r}'_u(u,v) = (-R\sin u \sin v \, , \; R\cos u \sin v \, , \; 0)$$

$$\bar{r}'_v(u,v) = (R\cos u \cos v \, , \; R\sin u \cos v \, , \; -R\sin v)$$

$$\rightsquigarrow \; \bar{n}(u,v) = \bar{r}'_u(u,v) \times \bar{r}'_v(u,v) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -R\sin u \sin v \, , & R\cos u \sin v \, , & 0 \\ R\cos u \cos v \, , & R\sin u \cos v \, , & -R\sin v \end{vmatrix}$$

$$= (-R^2 \cos u \sin^2 v)\,\bar{i} + (-R^2 \sin u \sin^2 v) + \bar{k}\,(-R^2 \sin v \cos v)$$

$$\Rightarrow \; \bar{n}(u,v) = \pm \left(-R^2 \cos u \sin^2 v \, , \; -R^2 \sin u \sin^2 v \, , \; -R^2 \sin v \cos v \right)$$

$$\rightsquigarrow \; \| \bar{n}(u,v) \| = \sqrt{R^4 \cos^2 u \sin^4 v + R^4 \sin^2 u \sin^4 v + R^4 \sin^2 v \cos^2 v}$$

$$= \sqrt{R^4 \sin^4 v + R^4 \sin^2 v \cos^2 v} = R^2 |\sin v| = R^2 \sin v$$

$$I = \iint_M R^2 \cos^2 u \sin^2 v \cdot R^2 \sin^2 u \sin^2 v \cdot R^2 \sin v \, du\, dv$$

$$M: \quad 0 \le u \le 2\pi \quad ; \quad 0 \le v \le \frac{\pi}{2}$$

$$I = R^6 \left(\int_0^{2\pi} \cos^2 u \sin^2 u \, du \right) \cdot \left(\int_0^{\pi/2} \sin^5 v \, dv \right)$$

$$\int_0^{2\pi} \cos^2 u \sin^2 u \, du = \frac{1}{4} \int_0^{2\pi} \sin^2 2u \, du = \frac{1}{4} \int_0^{2\pi} \frac{1 - \cos 4u}{2} \, du$$

$$= \frac{1}{8} \left[u - \frac{1}{4} \sin 4u \right]_0^{2\pi} = \frac{\pi}{4} \,//$$

$$\int_0^{\pi/2} \sin^5 v \, dv = \int_0^{\pi/2} \sin^4 v \cdot \sin v \, dv = \int_0^{\pi/2} (1 - \cos^2 v)^2 \sin v \, dv$$

$$= \left| \begin{array}{ll} t = -\cos v \, , & dt = \sin v \, dv \\ 0 \rightsquigarrow -1 \, , & \pi/2 \rightsquigarrow 0 \end{array} \right| =$$

$$= \int_{-1}^{0} (1 - t^2)^2 \, dt = \int_{-1}^{0} (1 - 2t^2 + t^4) \, dt = \left[t - \frac{2}{3} t^3 + \frac{t^5}{5} \right]_{-1}^{0}$$

$$= \frac{8}{15} \,//$$

$$\rightsquigarrow I = R^6 \cdot \frac{\pi}{4} \cdot \frac{8}{15} = \boxed{\frac{2\pi R^6}{15}}$$

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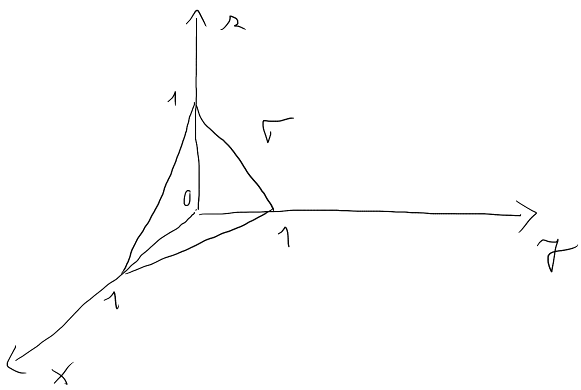
3. $I = \int\limits_{\Gamma} \frac{1}{(x+y+1)^2}\, dS$, Γ je hranica štvorstena :

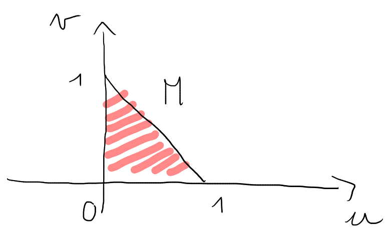
$x, y, z \geq 0$, $x + y + z \leq 1$



$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$

$I = I_1 + I_2 + I_3 + I_4$

a) Γ_1 : $y = 0$ ..... $x = u$, $z = v$, $u + v \leq 1$, $u, v \geq 0$



M_1 : $0 \leq u \leq 1$; $0 \leq v \leq 1 - u$

$$\bar{r}(u,v) = (u, 0, v) \rightsquigarrow \bar{r}_u' = (1,0,0), \quad \bar{r}_v = (0,0,1)$$

$$\bar{n}(u,v) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\bar{j} \Rightarrow \|\bar{n}(u,v)\| = 1$$

$$I_1 = \iint\limits_{M_1} \frac{1}{(u+1)^2}\, du\, dv = \int_0^1 \left[\int_0^{1-u} \frac{1}{(u+1)^2}\, dv \right] du =$$

$$= \int_0^1 \left[\frac{v}{(u+1)^2} \right]_0^{1-u} du = \int_0^1 \frac{1-u}{(u+1)^2}\, du =$$

$$= -\frac{1}{2} \int_0^1 \frac{2(u-1)}{(u+1)^2}\, du = -\frac{1}{2} \int_0^1 \frac{2(u+1)}{(u+1)^2}\, du + 2 \int_0^1 \frac{du}{(u+1)^2}$$

$$= \left[-\frac{1}{2} \ln (u+1)^2 - \frac{2}{u+1} \right]_0^1 = \left[-\ln |u+1| - \frac{2}{u+1} \right]_0^1$$

$$= -\ln 2 - 1 + 2 = 1 - \ln 2$$

b) $\sigma_2 : \quad x = 0 \quad \ldots \quad y = u, \ z = v \quad , \quad u+v \leq 1 \ , \ u+v \geq 0$

$$M_2 : \quad 0 \leq u \leq 1 \quad , \quad 0 \leq v \leq 1-u$$

$$\bar{r}(u,v) = (0, u, v) \rightsquigarrow \bar{r}_u'(u,v) = (0,1,0)$$

$$\bar{r}_v'(u,v) = (0,0,1)$$

$$\bar{n}(u,v) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \bar{i} \rightsquigarrow \| \bar{n}(u,v) \| = 1$$

$$I_2 = \iint_{M_2} \frac{1}{(u+1)^2} \, du \, dv = \underline{\underline{1 - \ln 2}} \quad (\text{rovnaké ako v a)})$$

c) $\Gamma_3 : \quad z = 0 \quad \ldots \quad x = u, \ y = v \quad ; \quad u+v \leq 1 \quad ; \quad u, v \geq 0$

$$M_3 : \quad 0 \leq u \leq 1 \quad ; \quad 0 \leq v \leq 1-u$$

$$\bar{r}(u,v) = (u, v, 0) \rightsquigarrow \bar{r}_u'(u,v) = (1, 0, 0)$$

$$\bar{r}_v'(u,v) = (0, 1, 0)$$

$$\bar{n}(u,v) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \bar{k} \rightsquigarrow \| \bar{n}(u,v) \| = 1$$

$$I_3 = \iint\limits_{M_3} \frac{1}{(u+v+1)^2}\,du\,dv = \int_0^1 \left[\int_0^{1-u} \frac{1}{(u+v+1)^2}\,dv \right] du$$

$$= \int_0^1 \left[-\frac{1}{u+v+1} \right]_0^{1-u} du = \int_0^1 \left(-\frac{1}{2} + \frac{1}{u+1} \right) du$$

$$= \left[-\frac{u}{2} + \ln|u+1| \right]_0^1 = -\frac{1}{2} + \ln 2$$

d) Γ_4 : $x+y+z=1$; $x=u$, $y=v$, $z=1-u-v$

M_4 : $0 \le u \le 1$, $0 \le v \le 1-u$

$\bar{r}(u,v) = (u, v, 1-u-v)$; $\bar{r}'_u = (1, 0, -1)$

$\bar{r}'_v = (0, 1, -1)$

$$\bar{n}(u,v) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \bar{i} + \bar{j} + \bar{k} \Rightarrow \|\bar{n}(u,v)\| = \sqrt{3}$$

$$I_4 = \iint\limits_{M_4} \frac{1}{(u+v+1)^2}\sqrt{3}\,du\,dv = \underline{\underline{\sqrt{3}\left(-\frac{1}{2} + \ln 2\right)}}$$ (podobně jako v c))

$$\rightsquigarrow I = I_1 + I_2 + I_3 + I_4 = 2(1-\ln 2) + (1+\sqrt{3})\left(-\frac{1}{2} + \ln 2\right)$$

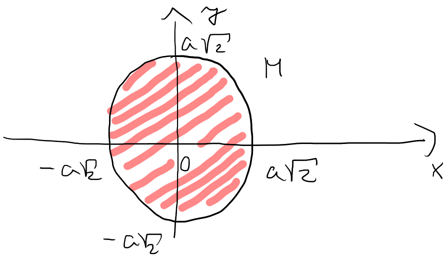
$$= \boxed{(\sqrt{3}-1)\ln 2 + \frac{3-\sqrt{3}}{2}}$$

4. APLIKÁCIE PLOŠNÉHO INTEGRÁLU I. DRUHU

MIERA PLOCHY σ :
$$S(\sigma) = \int_{\sigma} dS$$

$\rightsquigarrow$ miera časti paraboloidu $z = \frac{x^2+y^2}{2}$, $z \le a^2$, $a > 0$

$\rightsquigarrow$ parametrizácia : $x = u$, $y = v$, $z = \frac{u^2+v^2}{2}$



$$\frac{x^2+y^2}{2} = a^2 \rightsquigarrow x^2+y^2 = 2a^2$$

$$M : \quad u^2+v^2 \le 2a^2$$

$$\bar{r}(u,v) = \left(u, v, \frac{u^2+v^2}{2}\right) \rightsquigarrow \bar{r}'_u = (1, 0, u)$$
$$\bar{r}'_v = (0, 1, v)$$

$$\Rightarrow \bar{n}(u,v) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & u \\ 0 & 1 & v \end{vmatrix} = -u\bar{i} - v\bar{j} + \bar{k}$$

$$\rightsquigarrow \|\bar{n}(u,v)\| = \sqrt{u^2+v^2+1}$$

teda: $$S(\sigma) = \iint_M \|\bar{n}(u,v)\|\,du\,dv = \iint_M \sqrt{u^2+v^2+1}\,du\,dv ,$$

$$M : \quad u^2+v^2 \le 2a^2$$

$\rightsquigarrow$ polárne súradnice : $u = \rho\cos\varphi$, $v = \rho\sin\varphi$, $J = \rho$

$$M : \quad 0 \le \varphi \le 2\pi , \quad 0 \le \rho \le a\sqrt{2}$$

$$S(\sigma) = \int_0^{2\pi} \left[\int_0^{a\sqrt{2}} \sqrt{\rho^2+1}\ \rho\, d\rho \right] d\varphi = 2\pi \int_0^{a\sqrt{2}} \sqrt{\rho^2+1}\ \rho\, d\rho$$

$$= \left| \begin{array}{ll} t = \rho^2+1 \ , & dt = 2\rho\, d\rho \\ 0 \rightsquigarrow 1 \ ; & a\sqrt{2} \rightsquigarrow 2a^2+1 \end{array} \right| = \pi \int_1^{2a^2+1} \sqrt{t}\, dt = \pi \left[\frac{2t^{3/2}}{3} \right]_1^{2a^2+1}$$

$$= \pi \left[\frac{2}{3} (2a^2+1)^{3/2} - \frac{2}{3} \right] = \boxed{\frac{2}{3}\pi \left[(2a^2+1)^{3/2} - 1 \right]}$$

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5. obsah časti plochy σ s parametrickým vyjadrením :

$$\bar{r}(u,v) = (u\cos v,\ u\sin v,\ u^2)\ , \quad \left. \begin{array}{l} 0 \le u \le \sqrt{2} \\ 0 \le v \le \frac{\pi}{4} \end{array} \right\} : M$$

$$\rightsquigarrow \quad \bar{r}'_u = (\cos v,\ \sin v,\ 2u)$$

$$\bar{r}'_v = (-u\sin v,\ u\cos v,\ 0)$$

$$\Rightarrow \quad \bar{n}(u,v) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = -2u^2\cos v\, \bar{i} - 2u^2 \sin v\, \bar{j} + u\bar{k}$$

$$\Rightarrow \quad \|\bar{n}(u,v)\| = \sqrt{4u^4\cos^2 v + 4u^4\sin^2 v + u^2} = \sqrt{4u^4+u^2}$$

$$\dots \quad S(\sigma) = \iint_M \sqrt{4u^4+u^2}\, du\, dv = \int_0^{\pi/4} \left[\int_0^{\sqrt{2}} \sqrt{4u^4+u^2}\, du \right] dv$$

$$= \frac{\pi}{4} \int_0^{\sqrt{2}} \sqrt{4u^4 + u^2}\, du = \frac{\pi}{4} \int_0^{\sqrt{2}} \sqrt{4u^2+1} \cdot |u|\, du = \frac{\pi}{4} \int_0^{\sqrt{2}} \sqrt{4u^2+1}\, u\, du$$

$$= \left| \begin{array}{ll} t = 4u^2+1, & dt = 8u\,du \\ 0 \rightsquigarrow 1, & \sqrt{2} \rightsquigarrow 9 \end{array} \right| = \frac{\pi}{32} \int_1^9 \sqrt{t}\, dt = \frac{\pi}{32} \left[\frac{2}{3} t^{3/2} \right]_1^9$$

$$= \frac{\pi}{32} \left(\frac{2}{3} \cdot 27 - \frac{2}{3} \right) = \boxed{\frac{13\pi}{24}}$$