PLOŠNÉ INTEGRÁLY II. DRUHU

$I = \int_\sigma \bar{f} \cdot d\bar{S}$; $\bar{f}$ vektorová funkcia definovaná na orientovanej ploche σ → orientácia pomocou normálového vektora $\bar{n}(u,v) = \pm(\bar{r}'_u(u,v) \times \bar{r}'_v(u,v))$

→ prevod na dvojný integrál pomocou parametrizácie $\bar{r}(u,v)$:

$$I = \pm \iint_M \bar{f}(\bar{r}(u,v)) \cdot [\bar{r}'_u(u,v) \times \bar{r}'_v(u,v)]\, du\, dv$$

→ orientácia parametrizácie $\bar{r}(u,v)$:

→ ak $\bar{n} = +\bar{r}'_u \times \bar{r}'_v$ → súhlasná

→ ak $\bar{n} = -\bar{r}'_u \times \bar{r}'_v$ → nesúhlasná

→ klasický zápis:

$f = (f_1, f_2, f_3)$, $d\bar{S} = (dy\,dz, dx\,dz, dx\,dy)$

$$I = \int_\sigma f_1\, dy\,dz + f_2\, dx\,dz + f_3\, dx\,dy$$

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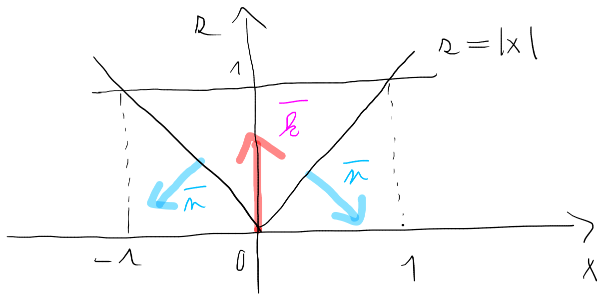
6. $I = \int_\sigma x^2\, dy\,dz + z^2\, dx\,dy$; $\sigma: x^2 + y^2 = z^2$, $0 \le z \le 1$, orientovaná tak, že $\measuredangle\, \bar{n}, \bar{k}$ je tupý

$\rightsquigarrow$ $f(x,y,z) = (x^2, 0, z^2)$; $\bar{r}(u,v) = (u, v, \sqrt{u^2+v^2})$

$M: \ u^2+v^2 \leq 1$



$\rightsquigarrow$ $\bar{r}'_u = \left(1, 0, \frac{u}{\sqrt{u^2+v^2}}\right)$, $\bar{r}'_v = \left(0, 1, \frac{v}{\sqrt{u^2+v^2}}\right)$

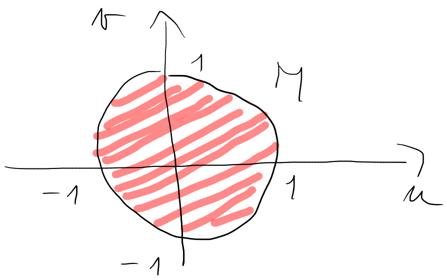
$$\bar{r}'_u \times \bar{r}'_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} = -\frac{u}{\sqrt{u^2+v^2}}\bar{i} - \frac{v}{\sqrt{u^2+v^2}} + \bar{k}$$

$$\Rightarrow \bar{n}(u,v) = -\frac{u}{\sqrt{u^2+v^2}}\bar{i} - \frac{v}{\sqrt{u^2+v^2}}\bar{j} + \bar{k}$$

$\rightsquigarrow$ $\bar{n}(u,v)\cdot\bar{k} = \bar{k}\cdot\bar{k} = 1 > 0 \Rightarrow$ parametrizácia je nesúhlasná s orientáciou

$$I = -\iint_M (u^2, 0, u^2+v^2)\cdot\left(-\frac{u}{\sqrt{u^2+v^2}}, -\frac{v}{\sqrt{u^2+v^2}}, 1\right) du\,dv$$

$$= \iint_M \left(\frac{u^3}{\sqrt{u^2+v^2}} - u^2 - v^2\right) du\,dv \ ;$$



$\leadsto$ polárne súradnice:

$$u = \rho\cos\varphi \;, \quad v = \rho\sin\varphi \;; \qquad 0 \le \varphi \le 2\pi$$
$$0 \le \rho \le 1$$

$$I = \int_0^{2\pi}\left[\int_0^1 \left(\frac{\rho^3\cos^3\varphi}{\rho} - \rho^2\right)\rho\, d\rho\right] d\varphi =$$

$$= \left(\int_0^{2\pi}(\cos^3\varphi - 1)\, d\varphi\right)\cdot\left(\int_0^1 \rho^3\, d\rho\right) = \left[\frac{\rho^4}{4}\right]_0^1 \cdot \int_0^{2\pi}(\cos^3\varphi - 1)\, d\varphi$$

$$= \frac{1}{4}\cdot\int_0^{2\pi}(\cos^3\varphi - 1)\, d\varphi = \frac{1}{4}\left[\int_0^{2\pi}\cos^3\varphi\, d\varphi - 2\pi\right] =$$

$$= -\frac{\pi}{2} + \frac{1}{4}\int_0^{2\pi}\cos^3\varphi\, d\varphi = -\frac{\pi}{2} + \frac{1}{4}\int_{-\pi}^{\pi}\cos^3\varphi\, d\varphi \quad \text{párna f-cia}$$

$$= -\frac{\pi}{2} + \frac{1}{4}\cdot 2\int_0^{\pi}\cos^3\varphi\, d\varphi$$

$\leadsto$ platí: $\cos^3\varphi = \frac{3}{4}\cos\varphi + \frac{1}{4}\cos 3\varphi$ //

$$\Rightarrow \text{teda} \int_0^{\pi}\cos^3\varphi\, d\varphi = \int_0^{\pi}\left[\frac{3}{4}\cos\varphi + \frac{1}{4}\cos 3\varphi\right]d\varphi = \left[\frac{3}{4}\sin\varphi + \frac{1}{12}\sin 3\varphi\right]_0^{\pi}$$

$$= 0 \;//$$

$$\Rightarrow \boxed{I = -\frac{\pi}{2}}$$

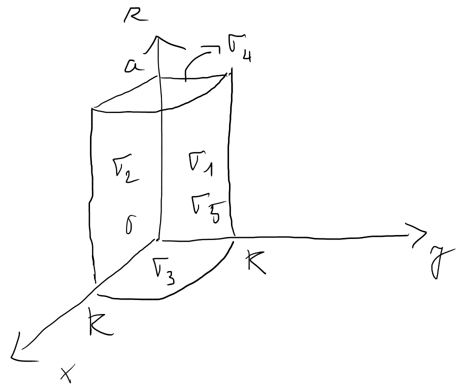
(7.)

$$I = \int_\sigma xz\,dy\,dz + xy\,dx\,dz + yz\,dx\,dy$$

σ : uzavretá plocha ako hranica oblasti v $\mathbb{R}^3$:

$x^2+y^2 \le R^2$, $x, y \ge 0$, $0 \le z \le a$, $R, a > 0$;

orientovaná normálou dnu



$\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4 \cup \sigma_5$

$I = I_1 + I_2 + I_3 + I_4 + I_5$

$f(x,y,z) = (xz, xy, yz)$

a) σ_1 : $x = 0$, $0 \le y \le R$; $0 \le z \le a$

$y = u$, $z = v$; $\bar{r}(u,v) = (0, u, v)$

M: $0 \le u \le R$, $0 \le v \le a$

$\bar{r}'_u = (0,1,0)$, $\bar{r}'_v = (0,0,1)$, predpísaná orientácia v smere $+\bar{i}$

$\bar{r}'_u \times \bar{r}'_v = \bar{i}$ $\Rightarrow$ parametrizácia súhlasná s orient.

$$I_1 = +\iint_M (0, 0, uv) \cdot (1,0,0)\,du\,dv = 0$$

b) σ_2: $y = 0$, $0 \le x \le R$, $0 \le z \le a$

$x = u$, $z = v$; $\bar{r}(u,v) = (u, 0, v)$

M: $0 \le u \le R$, $0 \le v \le a$

$\bar{r}'_u = (1, 0, 0)$, $\bar{r}'_v = (0, 0, 1)$, predpísaná orientácia v smere $+\bar{j}$

$\bar{r}'_u \times \bar{r}'_v = -\bar{j}$ $\Rightarrow$ parametrizácia nesúhlasná s orient.

$$I_2 = -\iint_M (uv, 0, 0) \cdot (0, -1, 0)\, du\, dv = 0$$

c) $z = 0$, $x^2 + y^2 \le R^2$ $\leadsto$ $x = u\cos v$, $y = u\sin v$

$\bar{r}(u,v) = (u\cos v, u\sin v, 0)$; M: $0 \le u \le R$
$0 \le v \le \frac{\pi}{2}$

$\leadsto$ predpísaná orientácia v smere $+\bar{k}$

$\bar{r}'_u = (\cos v, \sin v, 0)$, $\bar{r}'_v = (-u\sin v, u\cos v, 0)$

$$\bar{n}(u,v) = \bar{r}'_u \times \bar{r}'_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} =$$

$= u\bar{k}$; $u \ge 0$ $\Rightarrow$ parametrizácia súhlasná s orientáciou

$$I_3 = + \iint_M (0, u^2 \sin v \cos v, 0) \cdot (0,0,u)\, du\, dv = 0$$

d) Γ_4 : $z = a$, $x^2+y^2 \leq R^2$

situácia analogická ako v c) $\rightsquigarrow$

$$\bar{r}(u,v) = (u\cos v, u \sin v, a) , \quad M: \begin{array}{l} 0 \leq u \leq R \\ 0 \leq v \leq \frac{\pi}{2} \end{array}$$

$\rightsquigarrow$ predpísaná orientácia v smere $-\bar{k}$

$$\bar{r}'_u = (\cos v, \sin v) , \quad \bar{r}'_v = (-u \sin v, u \cos v, 0)$$

$\bar{n}(u,v) = \bar{r}'_u \times \bar{r}'_v = u\bar{k}$ $\rightsquigarrow$ parametrizácia nesúhlasná s orientáciou

$$I_4 = - \iint_M (au\cos v, u^2 \sin v \cos v, au \sin v) \cdot (0,0,u)\, du\, dv$$

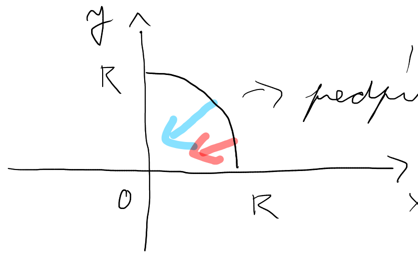
$$= - \iint_M a u^2 \sin v\, du\, dv = -a \left(\int_0^{\pi/2} \sin v\, dv \right) \cdot \left(\int_0^R u^2\, du \right) =$$

$$= -a \left[-\cos v \right]_0^{\pi/2} \cdot \left[\frac{u^3}{3} \right]_0^R = -\frac{aR^3}{3}$$

e) Γ_5 : $x^2+y^2 = R^2$, $x,y \geq 0$, $0 \leq z \leq a$

$x = R\cos v$, $y = R\sin v$, $z = u$, $M:$ $0 \le v \le \frac{\pi}{2}$

$0 \le u \le a$



$\to$ predpísaná orientácia

$\bar{r}(u,v) = (R\cos v, R\sin v, u)$

$\bar{r}'_u = (0,0,1)$, $\bar{r}'_v = (-R\sin v, R\cos v, 0)$

$$\bar{n}(u,v) = \bar{r}'_u \times \bar{r}'_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 0 & 1 \\ -R\sin v & R\cos v & 0 \end{vmatrix}$$

$$= (-R\cos v)\bar{i} - (R\sin v)\bar{j} =$$

$$= (-R\cos v)\bar{i} + (-R\sin v)\bar{j}$$ $\rightsquigarrow$ parametrizácia súhlasná s orientáciou

$$I_5 = +\iint_M (uR\cos v, R^2\sin v\cos v, uR\sin v)\cdot(-R\cos v, -R\sin v, 0)\, du\, dv$$

$$= \iint_M (-uR^2\cos^2 v - R^3\sin^2 v\cos v)\, du\, dv$$

$$= \int_0^{\pi/2}\left[\int_0^a (-uR^2\cos^2 v - R^3\sin^2 v\cos v)\, du\right] dv$$

$$= \int_0^{\pi/2} \left[-\frac{u^2}{2} k^2 \cos^2 v - u k^3 \sin^2 v \cos v \right]_0^a dv$$

$$= \int_0^{\pi/2} \left(-\frac{a^2 k^2}{2} \cos^2 v - a k^3 \sin^2 v \cos v \right) dv$$

$$= \int_0^{\pi/2} \left(-\frac{a^2 k^2}{4} - \frac{a^2 k^2}{4} \cos 2v - a k^3 \sin^2 v \cos v \right) dv$$

$$= \left[-\frac{a^2 k^2}{4} v - \frac{a^2 k^2}{8} \sin 2v \right]_0^{\pi/2} - a k^3 \int_0^{\pi/2} \sin^2 v \cos v \, dv$$

$$= -\frac{a^2 k^2 \pi}{8} - a k^3 \int_0^{\pi/2} \sin^2 v \cos v \, dv = \left| \begin{array}{l} t = \sin v \\ dt = \cos v \, dv \\ 0 \rightarrow 0 \ , \ \pi/2 \rightarrow 1 \end{array} \right|$$

$$= -\frac{a^2 k^2 \pi}{8} - a k^3 \int_0^1 t^2 \, dt = -\frac{a^2 k^2 \pi}{8} - a k^3 \left[t^3/3 \right]_0^1 =$$

$$= -\frac{a^2 k^2 \pi}{8} - \frac{a k^3}{3}$$

$$\Rightarrow I = I_1 + I_2 + I_3 + I_4 + I_5 = -\frac{a k^3}{3} - \frac{a^2 k^2 \pi}{8} - \frac{a k^3}{3}$$

$$= \boxed{-a k^2 \left(\frac{2}{3} k + \frac{\pi a}{8} \right)}$$

8. $I = \int_{\sigma} \bar{f} \cdot d\bar{s}$; $f(x,y,z) = (x^2, y^2, z^2)$

σ : časť jednotkovej sféry $S([0,0,0],1)$ s $x,y,z \geq 0$ (v I. oktante), ktorá je orientovaná normálou von //

$\rightsquigarrow$ $\bar{r}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$

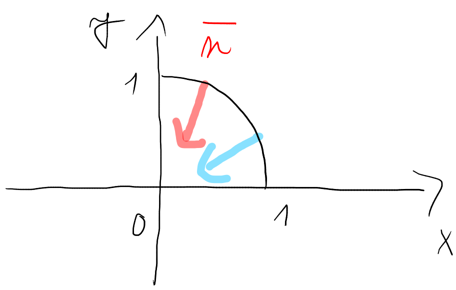
$M: 0 \leq u \leq \pi/2 \,, \quad 0 \leq v \leq \pi/2$

$\bar{r}'_u = (-\sin u \sin v, \cos u \sin v, 0)$

$\bar{r}'_v = (\cos u \cos v, \sin u \cos v, -\sin v)$

$$\bar{n}(u,v) = \bar{r}'_u \times \bar{r}'_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\sin u \sin v, & \cos u \sin v, & 0 \\ \cos u \cos v, & \sin u \cos v, & -\sin v \end{vmatrix}$$

$$= (-\cos u \sin^2 v)\,\bar{i} + (-\sin u \sin^2 v)\,\bar{j} + (-\sin v \cos v)\,\bar{k}$$



$\Rightarrow$ parametrizácia je súhlasná s orientáciou //

$$I = + \iint_M (\cos^2 u \sin^2 v, \sin^2 u \sin^2 v, \cos^2 v) \cdot$$
$$\cdot (-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v) \, du\, dv$$

$$= \iint_M \left(-\cos^3 u \sin^4 v - \sin^3 u \sin^4 v - \sin v \cos^3 v\right) du\, dv$$

$$= \int_0^{\pi/2} \left[\int_0^{\pi/2} \left[-\cos^3 u \sin^4 v - \sin^3 u \sin^4 v - \sin v \cos^3 v \right] du \right] dv$$

$$= \int_0^{\pi/2} \left[-\sin^4 v \int_0^{\pi/2} \cos^3 u\, du - \sin^4 v \int_0^{\pi/2} \sin^3 u\, du - \sin v \cos^3 v \left[u\right]_0^{\frac{\pi}{2}} \right] dv$$

$$= -\left(\int_0^{\pi/2} \cos^3 u\, du\right) \cdot \left(\int_0^{\pi/2} \sin^4 v\, dv\right) - \left(\int_0^{\pi/2} \sin^3 u\, du\right) \cdot \left(\int_0^{\pi/2} \sin^4 v\, dv\right)$$

$$-\frac{\pi}{2}\left(\int_0^{\pi/2} \cos^3 v \sin v\, dv\right)$$

- $$\int_0^{\pi/2} \cos^3 u\, du = \left| \begin{matrix} t = \sin u, & dt = \cos u\, du \\ 0 \rightsquigarrow 0, & \pi/2 \rightsquigarrow 1 \end{matrix} \right| = \int_0^1 (1-t^2) \cdot dt$$

$$= \left[t - t^3/3\right]_0^1 = 2/3$$

- $$\int_0^{\pi/2} \sin^3 u\, du = \left| \begin{matrix} t = \cos u, & dt = -\sin u\, du \\ 0 \rightsquigarrow 1; & \pi/2 \rightsquigarrow 0 \end{matrix} \right| = \int_0^1 (1-t^2)\, dt$$

$$= \left[t - t^3/3\right]_0^1 = 2/3$$

- $$\int_0^{\pi/2} \sin^4 v\, dv = \int_0^{\pi/2} \left(\frac{1-\cos 2v}{2}\right)^2 dv = \int_0^{\pi/2} \left(\frac{1}{4} - \frac{1}{2}\cos 2v + \frac{1}{4}\cos^2 2v\right) dv$$

$$= \int_0^{\pi/2} \left(\frac{1}{4} - \frac{1}{2}\cos 2v + \frac{1}{4}\left(\frac{1+\cos 4v}{2}\right)\right) dv =$$

$$= \int_0^{\pi/2} \left(\frac{1}{4} - \frac{1}{2}\cos 2v + \frac{1}{8} + \frac{1}{8}\cos 4v \right) dv = \int_0^{\pi/2} \left(\frac{3}{8} - \frac{1}{2}\cos 2v + \frac{1}{8}\cos 4v \right) dv$$

$$= \left[\frac{3}{8} v - \frac{1}{4}\sin 2v + \frac{1}{32}\sin 4v \right]_0^{\pi/2} = \frac{3\pi}{16}$$

- $$\int_0^{\pi/2} \cos^3 v \sin v \, dv = \left| \begin{array}{ll} t = \cos v, & dt = -\sin v \, dv \\ 0 \rightsquigarrow 1, & \pi/2 \rightsquigarrow 0 \end{array} \right|$$

$$= \int_0^1 t^3 \, dt = \left[\frac{t^4}{4} \right]_0^1 = \frac{1}{4}$$

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následne:

$$I = -\frac{2}{3} \cdot \frac{3\pi}{16} - \frac{2}{3} \cdot \frac{3\pi}{16} - \frac{\pi}{2} \cdot \frac{1}{4} = \boxed{-\frac{3\pi}{8}}$$

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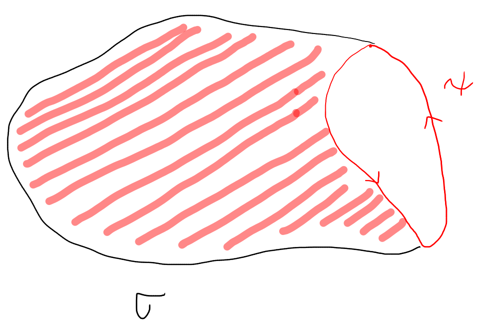
9. STOKESOVA VETA:



⇝ orientovaná plocha σ, ktorej okraj predstavuje orientovaná krivka γ

→ γ je orientovaná súhlasne s plochou σ

$$\oint_{\psi} \bar{f} \cdot d\bar{s} = \int_{\sigma} \operatorname{rot} \bar{f} \cdot d\bar{S}$$ ..... rot f je rotácia vektorového poľa f

ak $f = (P, Q, R)$, potom:

$$\operatorname{rot} f := \left(R'_y - Q'_z, P'_z - R'_x, Q'_x - P'_y \right)$$

⤳ pomocou Stokesovej vety vypočítajme krivkový integrál

$$I = \oint_{\psi} y\,dx + z\,dy + x\,dz$$; ψ je krivka daná ako prienik plôch $x^2 + y^2 + z^2 = a^2$, $x + y + z = 0$

$a > 0$

ψ je orientovaná takto:

$$\left[-\frac{a}{\sqrt{6}}, 0, \frac{a}{\sqrt{6}} \right] \to \left[0, -\frac{a}{\sqrt{6}}, \frac{a}{\sqrt{6}} \right] \to \left[\frac{a}{\sqrt{6}}, -\frac{a}{\sqrt{6}}, 0 \right]$$

⤳ ψ je kružnica so stredom v $[0, 0, 0]$ ležiaca v rovine $x + y + z = 0$

⤳ nech σ je časť tejto roviny, ktorá je ohraničená krivkou ψ, orientovaná tak, aby ψ bola orientovaná súhlasne so σ, t.j.:

σ je orientovaná v smere vektora $\bar{i} + \bar{j} + \bar{k}$

⤳ parametrizácia σ: $\bar{r}(u, v) = (u, v, -u - v)$, kde $[u, v] \in M$, M je priemet (kruhu) σ do roviny xy

⇝ platí:

$$\bar{r}'_u = (1,0,-1) \quad , \quad \bar{r}'_v = (0,1,-1)$$

$$\bar{n}(u,v) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \bar{i} + \bar{j} + \bar{k} \Rightarrow$$ parametrizácia je súhlasná so zvolenou orientáciou σ

⇝ platí: $f(x,y,z) = (y, z, x)$ a chceme vypočítať:

$$I = \oint_{\varphi} f \cdot d\bar{s}$$

⇝ podľa Stokesovej vety platí:

$$I = \int_{\sigma} \operatorname{rot} f \cdot d\bar{S}$$

⇝ keďže $P = y$, $Q = z$, $R = x$, máme:

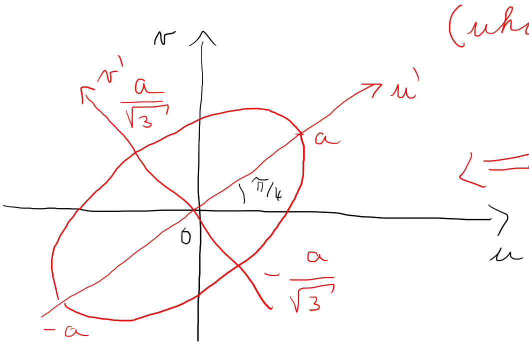
$$P'_y = 1 \,,\; P'_z = 0 \;;\; Q'_x = 0 \,,\; Q'_z = 1 \;;\; R'_x = 1 \,,\; R'_y = 0$$

$$\operatorname{rot} f(x,y,z) = (-1,-1,-1)$$

$$I = \iint_M (-1,-1,-1) \cdot \bar{n}(u,v)\,du\,dv = \iint_M (-1,-1,-1) \cdot (1,1,1)\,du\,dv$$

$$= -3 \iint_M du\,dv$$

→ M je elipsa v rovine xy so stredom v $[0,0]$ otočená o $\frac{\pi}{4}$ v kladnom smere ;



(uhol α medzi rovinami Γ a xy spĺňa : $\cos\alpha = \frac{1}{\sqrt{3}}$, lebo :

$$\cos\alpha = \frac{\bar{k} \cdot (\bar{i} + \bar{j} + \bar{k})}{\|\bar{k}\| \cdot \|\bar{i} + \bar{j} + \bar{k}\|} = \frac{1}{\sqrt{3}})$$

→ integrál $\iint\limits_M du\,dv$ je rovný obsahu danej červenej elipsy, t.j. :

$$\iint\limits_M du\,dv = \pi \cdot a \cdot \frac{a}{\sqrt{3}} = \frac{\pi a^2}{\sqrt{3}}$$

$$\Rightarrow \quad I = -3 \cdot \frac{\pi a^2}{\sqrt{3}} = \boxed{-\pi a^2 \sqrt{3}}$$

→ INÉ RIEŠENIE :

transformácia množiny M do polárnych súradníc v súradnico-vej sústave $u'v'$:

$$u = \left(a\varrho' \cos\varphi' - \frac{a}{\sqrt{3}}\varrho' \sin\varphi'\right)/\sqrt{2}$$

$$; \quad 0 \le \varrho' \le 1, \quad 0 \le \varphi' \le 2\pi$$

$$v = \left(a\varrho' \cos\varphi' + \frac{a}{\sqrt{3}}\varrho' \sin\varphi'\right)/\sqrt{2} \qquad J = \frac{a^2\varrho'}{\sqrt{3}}$$

$$\Rightarrow I = -3 \int_0^{2\pi} \left[\int_0^1 \frac{a^2 \varrho'}{\sqrt{3}}\, d\varrho' \right] d\varphi' = -a^2\sqrt{3} \left(\int_0^{2\pi} d\varphi'\right) \cdot \left(\int_0^1 \varrho'\, d\varrho'\right) = \underline{\underline{-a^2\pi\sqrt{3}}}$$

(10.) GAUSSOVA – OSTROGRADSKÉHO VETA

→ σ je častiach hladká Jordanova plocha orientovaná normálou von

→ $f(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ je spojité vektorové pole ; P'_x, Q'_y, R'_z sú spojité f-cie

$$\Rightarrow \quad I = \int_{\sigma} f \cdot d\bar{S} = \iiint_{M} \operatorname{div} f \, dx\, dy\, dz ,$$

kde M je vnútro plochy σ a $\operatorname{div} f := P'_x + Q'_y + R'_z$ je divergencia poľa f

→ pomocou Gaussovej – Ostrogradského vety vypočítajme plošný integrál :

$$I = \int_{\sigma} (x^2 \bar{i} + y^2 \bar{j} + z^2 \bar{k}) \cdot d\bar{S} ,$$

σ je hranica kocky $[0,a]^3$; $a > 0$, ktorá je orientovaná normálou von

→ $f(x,y,z) = (x^2, y^2, z^2)$ → $P = x^2$, $Q = y^2$, $R = z^2$

$P'_x = 2x$, $Q'_y = 2y$, $R'_z = 2z$ $\Rightarrow$ $\operatorname{div} f = 2(x+y+z)$

→ preto : $I = \iiint_{[0,a]^3} 2(x+y+z) \, dx\, dy\, dz$

$$= 2\int_0^a \left[\int_0^a \left[\int_0^a (x+y+z)\,dz \right] dy \right] dx =$$

$$= 2\int_0^a \left[\int_0^a \left[xz + yz + \frac{z^2}{2} \right]_0^a dy \right] dx = 2\int_0^a \left[\int_0^a \left(ax + ay + \frac{a^2}{2} \right) dy \right] dx$$

$$= 2\int_0^a \left[axy + \frac{ay^2}{2} + \frac{a^2}{2}y \right]_0^a dx = 2\int_0^a \left(a^2x + a^3 \right) dx$$

$$= 2\left[a^2\frac{x^2}{2} + a^3x \right]_0^a = \boxed{3a^4}$$