HARMONICKÝ RAD $\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\dots$ (D)

GEOMETRICKÝ RAD $\sum_{n=0}^{\infty} aq^n = a+aq+aq^2+\dots$

pre $|q|<1$ konverguje so súčtom $\frac{a}{1-q}$

pre $|q|\geq 1$ diverguje

$$s_n = a+aq+aq^2+aq^3+\dots+aq^{n-1}$$
$$\underline{q s_n = aq+aq^2+aq^3+aq^4+\dots+aq^n}$$

$$(1-q)s_n = a - aq^n \Rightarrow s_n = a\cdot\frac{1-q^n}{1-q} \ ; \ q\neq 1$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} a\cdot\frac{1-q^n}{1-q} = \begin{cases} \frac{a}{1-q} & ; \ |q|<1 \\ \infty & ; \ q>1 \\ \text{neexistuje} & ; \ q\leq -1 \end{cases}$$

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1. $\sum_{n=1}^{\infty} (-1)^n\cdot\left(\frac{5}{7}\right)^n = \sum_{n=1}^{\infty}\left(-\frac{5}{7}\right)^n \ ; \ q=-\frac{5}{7} \ ; \ |q|=\frac{5}{7}<1$

$\Rightarrow$ (K) $; \ s = \frac{-5/7}{1+5/7} = -\frac{5}{12}$

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2. $\sum_{n=1}^{\infty}\left(\frac{3}{4^{2n-1}}+\frac{2}{4^{2n}}\right) = \sum_{n=1}^{\infty}\frac{3}{4^{2n-1}} + \sum_{n=1}^{\infty}\frac{2}{4^{2n}} = \sum_{n=1}^{\infty} 12\cdot\left(\frac{1}{16}\right)^n +$

$+\sum_{n=1}^{\infty} 2\cdot\left(\frac{1}{16}\right)^n = \frac{12/16}{1-\frac{1}{16}} + \frac{\frac{2}{16}}{1-\frac{1}{16}} = \frac{\frac{14}{16}}{\frac{15}{16}} = \frac{14}{15}$

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3. $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$ $\qquad \frac{2n+1}{n^2(n+1)^2} = \frac{(n+1)^2-n^2}{n^2(n+1)^2} = \frac{1}{n^2}-\frac{1}{(n+1)^2}$

$$\sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)}_{\text{teleskopický rad}} = \left(1-\frac{1}{2^2}\right)+\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\left(\frac{1}{3^2}-\frac{1}{4^2}\right)+\dots+$$

$$s_n = 1 - \frac{1}{(n+1)^2} \quad ; \quad \lim_{n\to\infty} s_n = 1 \ \ldots \ K \quad ; \quad s = 1$$

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4. $$\sum_{n=1}^{\infty} \frac{2}{n(n+1)(n+2)} \quad ; \quad \frac{2}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$$

$$= \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) = \sum_{n=1}^{\infty}\left[\left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right]$$

$$= \underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{\text{teleskop.}} - \underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)}_{\text{teleskop.}}$$

$$s_n = 1 - \frac{1}{n+1} - \left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2} + \frac{1}{n+2} - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = \frac{1}{2} \Rightarrow K \quad ; \quad s = \frac{1}{2}$$

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5. $$\sum_{n=1}^{\infty} \frac{2n+1}{3^n} \quad ; \quad s_n = \frac{3}{3} + \frac{5}{3^2} + \frac{7}{3^3} + \frac{9}{3^4} + \ldots + \frac{2n+1}{3^n}$$

$$\frac{s_n}{3} = \frac{3}{3^2} + \frac{5}{3^3} + \frac{7}{3^4} + \frac{9}{3^5} + \ldots + \frac{2n+1}{3^{n+1}}$$

$$\frac{2}{3} s_n = 1 + \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \frac{2}{3^5} + \ldots + \frac{2}{3^n} - \frac{2n+1}{3^{n+1}}$$

$$\frac{2}{3} s_n = 1 + \frac{2}{3}\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^{n-1}}\right) - \frac{2n+1}{3^{n+1}}$$

$$s_n = \frac{3}{2} + \left(\frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^{n-1}}\right) - \frac{2n+1}{3^{n+1}}$$

$$s_n = \frac{3}{2} + \frac{1}{3} \cdot \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}} - \frac{2n+1}{3^{n+1}}$$

$$s_n = \frac{3}{2} + \frac{1}{2}\cdot\left(1-\frac{1}{3^n}\right) - \frac{2n+1}{3^{n+1}}$$

$$\lim_{n\to\infty} s_n = \frac{3}{2}+\frac{1}{2} = 2 \Rightarrow \text{K} \;;\; s=2$$

KRITÉRIÁ

NUTNÁ PODMIENKA KONVERG.

$$\sum a_n \;;\; \text{ak } \sum a_n \text{ K} \Rightarrow \lim_{n\to\infty} a_n = 0$$

6. $$\sum_{n=1}^{\infty} \ln n \;;\; \lim_{n\to\infty} \ln n = \infty \neq 0 \Rightarrow \text{D}$$

7. $$\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1} \;;\; \lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0 \Rightarrow \text{D}$$

8. $$\sum_{n=1}^{\infty} \frac{3+2\cdot(-1)^n n}{n+1} \;;\; \lim_{n\to\infty} \frac{3+2(-1)^n n}{n+1} = \text{neexistuje} \Rightarrow \text{D}$$

POROVNÁVACIE KRIT.

$$\sum a_n \;;\; \sum b_n : \quad a_n \geq b_n \,,\; \sum b_n \text{ D} \Rightarrow \sum a_n \text{ D}$$

$$a_n \leq b_n \,,\; \sum b_n \text{ K} \Rightarrow \sum a_n \text{ K}$$

9. $$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1} \;;\; a_n = \frac{n+1}{n^2+1} \geq \frac{1}{n} =: b_n$$

rad $\sum b_n = \sum \frac{1}{n}$ je D $\Rightarrow$ náš rad je D

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10. $$\sum_{n=1}^{\infty} \frac{1}{n^2} \;;\; a_n = \frac{1}{\underbrace{n^2}_{n\cdot n}} < \frac{1}{n(n-1)} \;;\; n \neq 1$$

$\sum b_n$; $b_n := \frac{1}{n(n-1)}$; $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \underbrace{\sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right)}_{\text{teleskop. suma}}$

$\sum b_n$ je (K) ; $a_n \leq b_n \Rightarrow$ náš rad je (K)

(11.) $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$; $\alpha \in [0,1] \cup [2,\infty)$

$\alpha = 0 \rightsquigarrow \sum 1 \rightsquigarrow$ (D)

$\alpha \in (0,1] \rightsquigarrow \frac{1}{n^{\alpha}} \geq \frac{1}{n} \rightsquigarrow$ (D)

$\alpha \in [2,\infty) \rightsquigarrow \frac{1}{n^{\alpha}} \leq \frac{1}{n^2} \rightsquigarrow$ (K)

(12.) $\sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{n+1}}$, $a_n = \frac{1}{n^2 \sqrt{n+1}}$; $b_n = \frac{1}{n^2}$ (K)

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2}{n^2 \sqrt{n+1}} = \lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = 0$$

$\Rightarrow$ náš rad je (K)

(13.) $\sum_{n=1}^{\infty} \sin \frac{\pi}{3^n}$, $a_n = \sin \frac{\pi}{3^n}$; nakoľko $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$b_n := \frac{\pi}{3^n}$, $\sum b_n$ je (K)

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\sin \frac{\pi}{3^n}}{\frac{\pi}{3^n}} = 1 \Rightarrow \text{náš rad (K)}$$

PODIELOVÉ KRITÉRIUM

$\frac{a_{n+1}}{a_n}$

nelimitná verzia:

ak $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow$ K

ak $\frac{a_{n+1}}{a_n} \geq 1$ pre nekonečne veľa indexov $\Rightarrow$ D

limitná verzia:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q \begin{cases} q<1 & \Rightarrow \text{K} \\ q>1 & \Rightarrow \text{D} \end{cases}$$

14. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$; $\frac{a_{n+1}}{a_n} = \frac{\frac{[(n+1)!]^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \frac{[(n+1)!]^2}{(n!)^2} \cdot \frac{(2n)!}{(2n+2)!}$

$$= (n+1)^2 \cdot \frac{1}{(2n+2)(2n+1)} = \frac{n+1}{2\cdot(2n+1)}, \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2(2n+1)} =$$

$$= \frac{1}{4} < 1 \Rightarrow \text{K}$$

15. $\sum_{n=1}^{\infty} \frac{n!}{2^n+1}$; $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{2^{n+1}+1}}{\frac{n!}{2^n+1}} = (n+1)\cdot\frac{2^n+1}{2^{n+1}+1}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (n+1)\frac{2^n+1}{2^{n+1}+1} = \lim_{n\to\infty} (n+1)\frac{1+\frac{1}{2^n}}{2+\frac{1}{2^n}} = \infty > 1$$

$\Rightarrow$ D

(16.) $\sum_{n=1}^{\infty} \frac{\varphi(n)}{\sqrt{6^n}}$; $\varphi(n) := \begin{cases} 2 & ; n \text{ je párne} \\ \sqrt{6} & ; n \text{ je nepárne} \end{cases}$

$$\frac{a_{n+1}}{a_n} = \frac{\varphi(n+1)}{\sqrt{6^{n+1}}} \cdot \frac{\sqrt{6^n}}{\varphi(n)} = \frac{\varphi(n+1)}{\varphi(n)} \cdot \frac{1}{\sqrt{6}} = \begin{cases} \frac{\sqrt{6}}{2} \cdot \frac{1}{\sqrt{6}} = \frac{1}{2} & ; n \text{ je pár.} \\ \frac{2}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} = \frac{1}{3} & ; n \text{ je nep.} \end{cases}$$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ neexistuje, ale $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2} < 1$

$\Rightarrow$ (K)

ODMOCNINOVÉ KRITÉRIUM

$\sqrt[n]{a_n}$; nelimitná verzia:

ak $\limsup_{n\to\infty} \sqrt[n]{(a_n)} < 1 \Rightarrow$ (K)

ak $\sqrt[n]{a_n} \geq 1$ pre nekonečne veľa $n \Rightarrow$ (D)

limitná verzia:

$\lim_{n\to\infty} \sqrt[n]{a_n} = q \begin{cases} q < 1 \Rightarrow (K) \\ q > 1 \Rightarrow (D) \end{cases}$

(17.) $\sum_{n=1}^{\infty} \left(\operatorname{arctg}\frac{1}{n}\right)^n$; $\sqrt[n]{a_n} = \operatorname{arctg}\frac{1}{n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \operatorname{arctg}\frac{1}{n} = 0 < 1 \Rightarrow (K)$$

(18.) $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$; $\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{n 2^n}} = \frac{1}{n^{1/n} \cdot 2}$

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{1}{n^{1/n}\cdot 2}=\frac{1}{2}<1\Rightarrow K$$

RAABEHO KRITERIUM

$$n\left(1-\frac{a_{n+1}}{a_n}\right)$$

19.
$$\sum_{n=1}^{\infty}\frac{1}{3^{1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{n}}}$$

$$\frac{a_{n+1}}{a_n}=\frac{\frac{1}{3^{1+1/2+\dots+1/(n+1)}}}{\frac{1}{3^{1+1/2+\dots+1/n}}}=\frac{1}{3^{\frac{1}{n+1}}}$$

$$n\left(1-\frac{a_{n+1}}{a_n}\right)=n\left(1-\frac{1}{3^{\frac{1}{n+1}}}\right)=$$

$$=\frac{n}{n+1}\cdot\frac{1}{3^{\frac{1}{n+1}}}\cdot\frac{3^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$$

$$\lim_{n\to\infty}n\left(1-\frac{a_{n+1}}{a_n}\right)=\lim_{n\to\infty}\left[\frac{n}{n+1}\cdot\frac{1}{3^{\frac{1}{n+1}}}\cdot\frac{3^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\right]=$$

$$=\ln 3>1\Rightarrow K$$