POSTUPNOSTI A RADY FUNKCIÍ – BODOVÁ A ROVNOMERNÁ KONVERGENCIA

$$\{f_n(x)\}_{n=1}^{\infty} \quad ; \quad \sum_{n=1}^{\infty} f_n(x) \quad ; \quad x \in I$$

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1. $f_n(x) = \operatorname{arctg} nx \quad , \quad x \in \mathbb{R}$

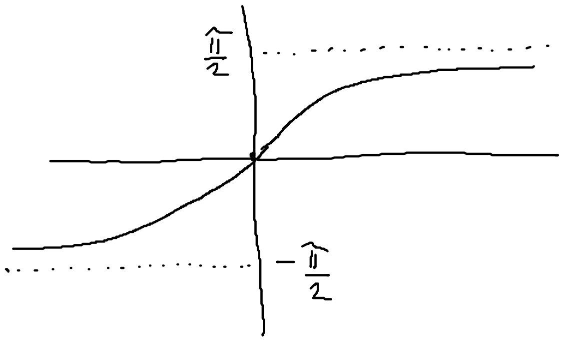
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \operatorname{arctg} nx = \begin{cases} -\frac{\pi}{2} & ; \ x<0 \\ 0 & ; \ x=0 \\ \frac{\pi}{2} & ; \ x>0 \end{cases} \Rightarrow f(x) = \begin{cases} -\pi/2 & ; \ x<0 \\ 0 & ; \ x=0 \\ \pi/2 & ; \ x>0 \end{cases}$$

$\underline{OK = \mathbb{R}}$, $f_n \to f$ na $\mathbb{R}$

rovnomerná konvergencia?

$$a_n := \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \quad ; \quad \lim_{n\to\infty} a_n = 0 \ \Rightarrow \ f_n \rightrightarrows f \text{ na } \mathbb{R}$$

$$a_n = \sup_{x\in\mathbb{R}} |\operatorname{arctg} nx - f(x)| = \sup_{x\in(0,\infty)} \left|\operatorname{arctg} nx - \frac{\pi}{2}\right| = \frac{\pi}{2} \ \Rightarrow \ \lim_{n\to\infty} a_n \neq 0$$



$f_n \not\rightrightarrows f$

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2. $f_n(x) = e^{-nx} \quad , \quad x \in \mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{-nx} = \begin{cases} \infty & , \ x<0 \\ 1 & , \ x=0 \\ 0 & , \ x>0 \end{cases} \Rightarrow OK = [0,\infty) \ , \ f(x) = \begin{cases} 1 & , x=0 \\ 0 & , x>0 \end{cases}$$

$$a_n = \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in(0,\infty)} |e^{-nx}| = 1 \ \Rightarrow \ \lim_{n\to\infty} a_n \neq 0 \ , \ f_n \not\rightrightarrows f$$

3. $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$, $x \in \mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0 \quad ; \quad f(x) \equiv 0 \ , \ OK = \mathbb{R}$$

$$a_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \frac{|\sin nx|}{\sqrt{n}} = \frac{1}{\sqrt{n}} \quad ; \quad \lim_{n\to\infty} a_n = 0$$

$$\rightsquigarrow f_n \rightrightarrows f$$

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4. $f_n(x) = x^n - x^{2n}$, $x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (x^n - x^{2n}) = 0 \rightsquigarrow f(x) \equiv 0 \ , \ OK = [0,1]$$

$$a_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |x^n - x^{2n}| = \max_{x\in[0,1]} (x^n - x^{2n})$$

$$(x^n - x^{2n})' = nx^{n-1} - 2nx^{2n-1} = 0 \rightsquigarrow x = \frac{1}{\sqrt[n]{2}} \in [0,1]$$

$$(x^n - x^{2n})\Big|_{x=\frac{1}{2^{1/n}}} = \frac{1}{2} \quad , \quad (x^n - x^{2n})\Big|_{x=0} = 0 = (x^n - x^{2n})\Big|_{x=1}$$

$$\Rightarrow a_n = \frac{1}{2} \quad a \quad \lim_{n\to\infty} a_n = \frac{1}{2} \neq 0 \rightsquigarrow f_n \not\rightrightarrows f$$

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5. $f_n(x) = \dfrac{x^n}{1+x^n}$, $x \in \mathbb{R}\setminus\{-1\}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \begin{cases} 0 , & x=0 \\ \lim\limits_{n\to\infty} \dfrac{1}{x^{-n}+1} = \begin{cases} 1 ; & x \neq 1,0 \\ \frac{1}{2} ; & x=1 \end{cases} \end{cases}$$

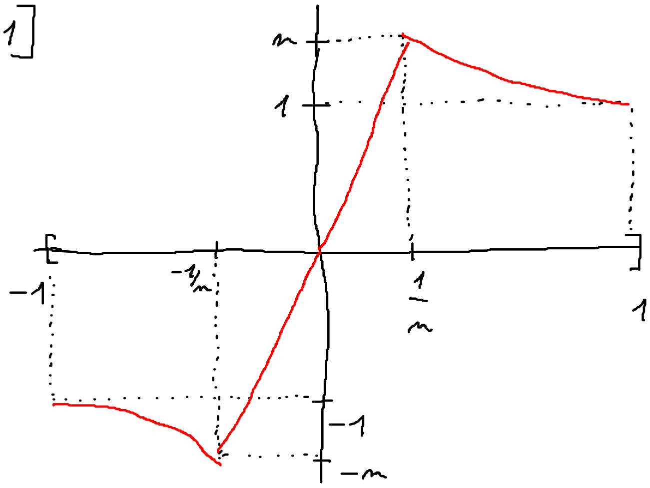
$$\rightsquigarrow f(x) = \begin{cases} 0 , & x=0 \\ 1/2 , & x=1 \\ 1 , & x \neq 0,1 \end{cases} \quad ; \quad OK = \mathbb{R}\setminus\{-1\}$$

$$a_n = \sup_{x\in\mathbb{R}\setminus\{-1\}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}\setminus\{\pm1,0\}} \left|\frac{x^n}{1+x^n} - 1\right| = \sup_{x\in\mathbb{R}\setminus\{\pm1,0\}} \frac{1}{|1+x^n|} = \frac{1}{2}$$

.... $\lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$ $\qquad f_n \not\rightrightarrows f$

(6.)

$$f_n(x) = \begin{cases} \frac{1}{x} &, x \in [-1, -1/n] \cup [\frac{1}{n}, 1] \\ n^2 x &, x \in (-\frac{1}{n}, \frac{1}{n}) \end{cases}$$



$$\lim_{n\to\infty} f_n(x) = \begin{cases} \frac{1}{x} &; [-1,1]\setminus\{0\} \\ 0 &; x = 0 \end{cases}$$

$$\rightsquigarrow f(x) = \begin{cases} 1/x &, x\in[-1,1]\setminus\{0\} \\ 0 &, x = 0 \end{cases} \qquad OK = [-1,1]$$

$$a_n = \sup_{x\in[-1,1]} |f_n(x) - f(x)| = \sup_{x\in(-\frac{1}{n},\frac{1}{n})\setminus\{0\}} \left|n^2 x - \frac{1}{x}\right| = \infty, \quad \lim_{n\to\infty} a_n \neq 0$$

$$f_n \not\rightrightarrows f$$

(7.)

$$\sum_{n=1}^{\infty} \left(\frac{x}{1+x}\right)^n, \quad x \in \mathbb{R}\setminus\{-1\}$$

$\rightsquigarrow$ geom. rad pre $\forall x$ s kvocientom $\frac{x}{1+x}$ $\Rightarrow$ $\left|\frac{x}{1+x}\right| < 1$

$\Rightarrow$ OK: $x \in (-\frac{1}{2}, \infty)$ $\qquad s(x) = \sum_{n=1}^{\infty} \left(\frac{x}{1+x}\right)^n = \frac{x}{1+x} \cdot \frac{1}{1 - \frac{x}{1+x}} = x$

(8.) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $x \in [-1,1]$

WEIERSTRASSOVO MAJORANTNÉ KRITÉRIUM ROVNOMERNEJ KONV.

$\left|\frac{x^n}{n^2}\right| = \frac{|x|^n}{n^2} \leq \frac{1}{n^2}$, číselný rad $\sum \frac{1}{n^2}$ (K)

$\Rightarrow \sum f_n \rightrightarrows s$ na $[-1,1]$

(9.) $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$, $x \in \mathbb{R}$

$\left|\frac{1}{n^2+x^2}\right| = \frac{1}{n^2+x^2} \leq \frac{1}{n^2}$; číselný rad $\sum \frac{1}{n^2}$ (K)

$\Rightarrow \sum f_n \rightrightarrows s$ na $\mathbb{R}$

(10.) $\sum_{n=1}^{\infty} \frac{1}{n^x}$, $x \in \mathbb{R}$ $\rightsquigarrow$ $OK = (1,\infty)$; platí, že daný rad konverguje lokálne rovnomerne na $(1,\infty)$

(ale nie rovnomerne na $(1,\infty)$)

(11.) $\sum_{n=1}^{\infty} x^n \operatorname{tg}\left(\frac{x}{2^n}\right)$, $x \in \mathbb{R}$

podielové krit.: $\left|\frac{x^{n+1} \operatorname{tg} \frac{x}{2^{n+1}}}{x^n \operatorname{tg} \frac{x}{2^n}}\right| = |x| \cdot \left|\frac{\operatorname{tg} \frac{x}{2^{n+1}}}{\operatorname{tg} \frac{x}{2^n}}\right| \xrightarrow{n\to\infty} \frac{|x|}{2}$

pre $|x|<2$ (K) , pre $|x|>2$ (D)

$|x| = 2 \quad \rightsquigarrow \quad x = 2 \; \dots \; \sum_{n=1}^{\infty} 2^n \cdot \operatorname{tg} \frac{2}{2^n} = \sum_{n=1}^{\infty} 2 \cdot \frac{\operatorname{tg} 2^{-n+1}}{2^{-n+1}}$

$\rightsquigarrow$ nie je splnená nutná podm. … D

$x = -2 \; \dots \; \sum_{n=1}^{\infty} (-2)^n \operatorname{tg} \frac{(-2)}{2^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\operatorname{tg} 2^{-n+1}}{2^{-n+1}} \cdot 2$

$\rightsquigarrow$ nie je splnená nutná podm. …. D

$\Rightarrow$ OK : $x \in (-2, 2)$

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12. $\sum_{n=1}^{\infty} \frac{\sin^2 nx}{\sqrt[3]{n^4 + x^4}}$, $x \in \mathbb{R}$

$$\left| \frac{\sin^2 nx}{\sqrt[3]{n^4 + x^4}} \right| = \frac{|\sin^2 nx|}{\sqrt[3]{n^4 + x^4}} \leq \frac{1}{\sqrt[3]{n^4}} \text{ ; rad } \sum \frac{1}{\sqrt[3]{n^4}} \text{ (K)}$$

$\Rightarrow \sum f_n \rightrightarrows s$ na $\mathbb{R}$

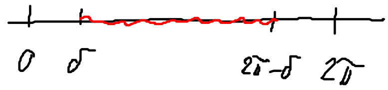
DIRICHLETOVO A ABELOVO KRITÉRIUM

$\sum_{n=1}^{\infty} f_n(x) g_n(x)$, $\{g_n(x)\}$ je monotónna postup. na I

DIRICHLET: ak $g_n \rightrightarrows 0$ na I a $\sum f_n$ má rovnomerne ohraničené čiast. súčty na I $\Rightarrow \sum f_n g_n \rightrightarrows$ na I

ABEL: ak $\{g_n\}$ je rovnomerne ohraničená na I a rad $\sum f_n \rightrightarrows$ na I $\Rightarrow \sum f_n g_n \rightrightarrows$ na I

13. $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$; $x \in I_\delta = [\delta, 2\pi - \delta]$, $\delta \in (0, \pi)$



$f_n(x) := \sin nx$, $g_n(x) := \frac{1}{n}$ $\{g_n(x)\}$ je monotónna na I_δ

- $\{g_n(x)\} \rightrightarrows 0$ na I_δ
- $\sum f_n(x) = \sum \sin nx$ má ohraničené čiast. súčty na I_δ

$\Rightarrow$ DIRICHLET $\Rightarrow \sum \frac{\sin nx}{n} \rightrightarrows$ na I_δ pre $\forall \delta \in (0, \pi)$

(teda náš rad (K) lokálne rovnomerne na $(0, 2\pi)$)

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14. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n(n+x)}}$; $x \in [0, \infty)$

$f_n(x) := \frac{(-1)^n}{n}$, $g_n(x) := \frac{1}{\sqrt{1+\frac{x}{n}}}$, $\{g_n(x)\}$ je monotónna na $[0, \infty)$

- $\{g_n(x)\}$ je rovnomerne ohraničená na $[0, \infty)$, lebo :

$$|g_n(x)| = \frac{1}{\sqrt{1+\frac{x}{n}}} \leq 1 \quad \text{pre } \forall x \in [0, \infty) \text{ a } \forall n \in \mathbb{N}$$

- rad $\sum f_n(x) = \sum \frac{(-1)^n}{n} \rightrightarrows$ na $[0, \infty)$ (nezávisí na x)

$\Rightarrow$ ABEL $\Rightarrow \sum \frac{(-1)^n}{\sqrt{n(n+x)}} \rightrightarrows$ na $[0, \infty)$

MOCNINOVÉ RADY

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \;,\; a_n \;,\; x_0 \;,\; R := \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} \rightsquigarrow \text{polomer konv.}$$

interval konvergencie $(x_0 - R, x_0 + R)$ | ak $\lim \left|\frac{a_{n+1}}{a_n}\right|$ existuje, potom: $R = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

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15. $$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n \;;\; a_n = \frac{(n!)^2}{(2n)!} \;,\; x_0 = 0$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2(2n+1)} \xrightarrow{n\to\infty} \frac{1}{4} \Rightarrow R = 4$$

IK: $(-4, 4)$

OBOR KONVERGENCIE?

$x = 4$ $\quad \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 4^n \rightsquigarrow$ D (RAABE)

$x = -4$ $\quad \sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!} 4^n \rightsquigarrow$ D (nie je splnená nutná podmienka konvergencie; postup. $\frac{(n!)^2}{(2n)!} 4^n$ je rastúca)

$\Rightarrow$ OK: $(-4, 4)$

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16. $$\sum_{n=1}^{\infty} \frac{x^{4n-3}}{4n-3} \;;\; x_0 = 0 \;,\; a_m = \begin{cases} 0 \;, & m \equiv 0, -1, -2 \pmod 4 \\ \frac{1}{m} \;, & m \equiv -3 \pmod 4 \end{cases}$$

$$\sqrt[m]{|a_m|} = \begin{cases} 0 , & m \equiv 0,-1,-2 \pmod 4 \\ \frac{1}{\sqrt[m]{m}} , & m \equiv -3 \pmod 4 \end{cases} \Rightarrow \limsup_{m\to\infty} \sqrt[m]{|a_m|} = 1$$

$\rightsquigarrow$ $R = 1$, IK: $(-1,1)$

$x=1$: $\sum_{n=1}^{\infty} \frac{1}{4n-3}$ ... D $\qquad$ $x=-1$: $\sum_{n=1}^{\infty} \frac{-1}{4n-3}$ ... D

OK: $(-1,1)$

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17. $\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n(n+1)(n+2)}$, $x_0 = -1$, $a_n = \frac{1}{2^n(n+1)(n+2)}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} \cdot \frac{n+1}{n+3} \xrightarrow{n\to\infty} \frac{1}{2} \Rightarrow R = 2$$

IK: $(-3, 1)$

$x=-3$: $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n(n+1)(n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$ .... K

$x=1$: $\sum_{n=0}^{\infty} \frac{2^n}{2^n(n+1)(n+2)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$ ..... K

OK: $[-3,1]$

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18. $\sum_{n=1}^{\infty} x^n \cdot \sin\frac{1}{n}$; $x_0 = 0$, $a_n = \sin\frac{1}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sin\frac{1}{n+1}}{\sin\frac{1}{n}}\right| \xrightarrow{n\to\infty} 1 \Rightarrow \underline{\underline{R = 1}}$$

IK: $\underline{\underline{(-1, 1)}}$

$x = -1$: $\sum_{n=1}^{\infty} (-1)^n \sin\frac{1}{n}$ .... K

$\rightsquigarrow$ OK: $\underline{\underline{[-1, 1)}}$

$x = 1$: $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ .... D

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19. $\sum_{n=1}^{\infty} n^2 x^n$, $x_0 = 0$, $a_n = n^2$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n}\right)^2 \xrightarrow{n\to\infty} 1 \rightsquigarrow R = 1 , \ IK = (-1, 1)$$

OK $= \underline{\underline{(-1, 1)}}$

<u>SÚČET NA $(-1,1)$?</u>

<u>geometrický rad</u> $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$; $x \in (-1, 1)$

$\rightsquigarrow$ derivujeme : $\sum_{n=1}^{\infty} n x^{n-1} = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}$

$\Downarrow$

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} , \quad x \in (-1, 1)$$

$\rightsquigarrow$ opäť derivujeme :

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\frac{x}{(1-x)^2}\right)' = \frac{x+1}{(1-x)^2}$$

$$\Downarrow$$

$$\boxed{\sum_{n=1}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^2} \quad , \quad x \in (-1,1)}$$

20.

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} \quad , \quad x_0 = 0 \quad , \quad a_n = \frac{1}{n(n+1)}$$

$\rightsquigarrow$ zistíme, že $R = 1$, $IK = (-1,1)$, $OK = [-1,1]$

SÚČET NA $[-1,1]$?

$\rightsquigarrow$ vieme súčet na $(-1,1)$:

$$\sum_{n=1}^{\infty} t^{n-1} = \frac{1}{1-t} \quad , \quad t \in (-1,1) \qquad \text{(geom. rad)}$$

$\rightsquigarrow$ integrujeme v medziach od 0 do $x \in (-1,1)$:

$$\sum_{n=1}^{\infty} \int_0^x t^{n-1}\,dt = \int_0^x \frac{dt}{1-t} \rightsquigarrow \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln|x-1| \, , \; x \in (-1,1)$$

$\rightsquigarrow$ získaný rad opäť integrujeme v medziach od 0 do $x \in (-1,1)$:

$$\sum_{n=1}^{\infty} \int_0^x \frac{t^n}{n}\,dt = -\int_0^x \ln|t-1|\,dt$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = x - (x-1)\ln|x-1|$$

$$\boxed{\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = 1 - \frac{(x-1)\ln|x-1|}{x} \quad , \ x \in (-1,1)}$$

$x = -1$

$$s(-1) = \lim_{x \to -1^+} s(x) = \lim_{x \to -1^+} \left(1 - \frac{(x-1)\ln|x-1|}{x}\right) = 1 - 2\ln 2$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = 1 - 2\ln 2}$$

$x = 1$ $\quad s(1) = \lim_{x \to 1^-} s(x) = \lim_{x \to 1^-} \left(1 - \frac{(x-1)\ln|x-1|}{x}\right) = 1$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1}$$

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21. $\displaystyle\sum_{n=1}^{\infty} \frac{n x^{n-1}}{2^n} \quad \overset{m = n-1}{\rightsquigarrow} \quad \sum_{m=0}^{\infty} \frac{(m+1)\cdot x^m}{2^{m+1}}$

$x_0 = 0$, $a_m = \dfrac{m+1}{2^{m+1}}$; rièdme :

$R = 2$, $IK = (-2,2)$, $\underline{\underline{OK = (-2,2)}}$

SÚČET ?

pre $x \in (-2,2)$ je $\frac{x}{2} \in (-1,1)$ a : $\displaystyle\sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^{m+1} = \frac{\frac{x}{2}}{1-\frac{x}{2}} = \frac{x}{2-x}$

(geomet. rad)

⇝ tento rad derivujeme :

$$\sum_{n=0}^{\infty} (n+1)\cdot \frac{x^n}{2^{n+1}} = \left(\frac{x}{2-x}\right)' = \frac{2}{(x-2)^2}$$

$$\Downarrow$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(n+1)x^n}{2^{n+1}} = \frac{2}{(x-2)^2} \quad , \quad x \in (-2,2)}$$

SÚČTY NIEKTORÝCH ČÍSELNÝCH RADOV

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22. $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ ..... K ⇝ súčet ???

⇝ použijeme vhodný mocninový rad ⇝ $\sum_{n=1}^{\infty} \frac{x^n}{n}$, pre $x = \frac{1}{3}$

rad $\sum_{n=1}^{\infty} \frac{x^n}{n}$ má OK $= [-1,1)$ a jeho súčet je :

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln|x-1| \quad , \quad x \in [-1,1)$$

⇝ potom :

$$\sum_{n=1}^{\infty} \frac{1}{n3^n} = \sum_{n=1}^{\infty} \frac{(1/3)^n}{n} = -\ln\left|\frac{1}{3}-1\right| = \ln\frac{3}{2}$$

(23.) $\sum_{n=1}^{\infty} \frac{n(n+1)}{8^n}$ .... (K) súčet ???

→ mocninový rad $\sum_{n=1}^{\infty} n(n+1)x^n$ s $x = 1/8$

→ tento rad má OK $= (-1, 1)$ a súčet:

$$\sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3} \quad , \; x \in (-1, 1)$$

→ pre $x = 1/8$ máme:

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{8^n} = \sum_{n=1}^{\infty} n(n+1)(1/8)^n = \frac{1/4}{(1-1/8)^3} = \frac{128}{343}$$
