TAYLOROV RAD

$f(x)$, x_0

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad ; \quad x \in \mathcal{O}(x_0)$$

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad ; \quad x \in \mathcal{O}(x_0)$$

24.

$$f(x) = \sqrt{1+x} \quad ; \quad x_0 = 0$$

$$f(x)\Big|_0 = 1$$

$$f'(x)\Big|_0 = \frac{1}{2}(1+x)^{-1/2}\Big|_0 = \frac{1}{2}$$

$$f''(x)\Big|_0 = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot(1+x)^{-\frac{3}{2}}\Big|_0 = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)$$

$$f'''(x)\Big|_0 = \frac{1}{2}\left(-\frac{1}{2}\right)\cdot\left(\frac{-3}{2}\right)\cdot(1+x)^{-\frac{5}{2}}\Big|_0 = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(\frac{-3}{2}\right)$$

$$f^{(4)}(x)\Big|_0 = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(\frac{-3}{2}\right)\cdot\left(\frac{-5}{2}\right)\cdot(1+x)^{-\frac{7}{2}}\Big|_0 = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(\frac{-3}{2}\right)\cdot\left(\frac{-5}{2}\right)$$

$$\rightsquigarrow f^{(n)}(0) = \frac{1}{2}\cdot\frac{(-1)^{n-1}\cdot(1\cdot 3\cdot 5\cdots[2(n-1)-1])}{2^{n-1}} = \frac{1}{2^n}\cdot(-1)^{n-1}(1\cdot 3\cdot 5\cdots(2n-3))$$

$$= \frac{1}{2^n}\cdot(-1)^{n-1}\cdot(2n-3)!! \quad ; \quad n \geq 2$$

$$\rightsquigarrow T(x) = 1 + \frac{x}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot(2n-3)!!}{2^n\cdot n!}x^n \quad ; \quad T(x) = f(x) \; ; \; x \in (-1,1)$$

25.

$$f(x) = e^{-x} \quad ; \quad x_0 = 0$$

vemo, če : $e^t = \sum_{n=0}^{\infty}\frac{t^n}{n!}$, $t \in \mathbb{R} \rightsquigarrow t := -x$

$$\rightsquigarrow T(x) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot x^n \quad ; \quad e^{-x} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n \; ; \; \forall x \in \mathbb{R}$$

26.

$$f(x) = \cos(x^2) \quad ; \quad x_0 = 0$$

vieme, že : $\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \quad , \quad t \in \mathbb{R}$

$\rightsquigarrow$ $t := x^2$ $\rightsquigarrow$ $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{4n} \quad ; \quad \forall x \in \mathbb{R}$

$\rightsquigarrow$ $T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} \quad ; \quad \forall x \in \mathbb{R}$

27.

$$f(x) = \frac{x-3}{(x+1)^2} \quad ; \quad x_0 = 0$$

$\rightsquigarrow$ $T(x) = ???$

$\rightsquigarrow$ platí : $\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x} \quad ; \quad x \in (-1,1)$

derivujeme

$$\sum_{n=1}^{\infty} n \cdot (-1)^n \cdot x^{n-1} = -\frac{1}{(x+1)^2}$$

$$\underline{\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n \cdot x^n = \frac{x}{(x+1)^2}} \quad , \quad x \in (-1,1)$$

$\rightsquigarrow$ platí : $\sum_{n=1}^{\infty} 3n(-1)^n \cdot x^{n-1} = \frac{-3}{(x+1)^2} \quad , \quad x \in (-1,1)$

$\rightsquigarrow$ preto f-cia $\frac{x-3}{(x+1)^2}$ má tvar :

$$\frac{x-3}{(x+1)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot n \cdot x^n - \sum_{n=1}^{\infty} 3 \cdot (-1)^n \cdot n x^{n-1} =$$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} m x^m - \sum_{m=0}^{\infty} 3 \cdot (-1)^{m+1} (m+1) x^m =$$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} m x^m - \sum_{m=0}^{\infty} 3 \cdot (-1)^{m-1} (m+1) x^m$$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \left[m - 3(m+1) \right] x^m + 3$$

$$= \sum_{m=1}^{\infty} (-1)^m \left[2m + 3 \right] x^m + 3 = \sum_{m=0}^{\infty} (-1)^m \left[2m+3 \right] x^m$$

$$\Rightarrow f(x) = \frac{x-3}{(x+1)^2} = \sum_{n=0}^{\infty} (-1)^n (2n+3) x^n \quad ; \quad x \in (-1,1)$$

$$\rightsquigarrow T(x) = \sum_{n=0}^{\infty} (-1)^n (2n+3) x^n \quad ; \quad x \in (-1,1)$$

TRIGONOMETRICKÝ FOURIEROV RAD

$$f(x) \; ; \; \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \; , \; n \in \mathbb{N}_0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \; , \; n \in \mathbb{N}$$

$\rightsquigarrow$ DIRICHLETOVA VETA – ak f je po častiach monotónna a po častiach spojitá, potom:

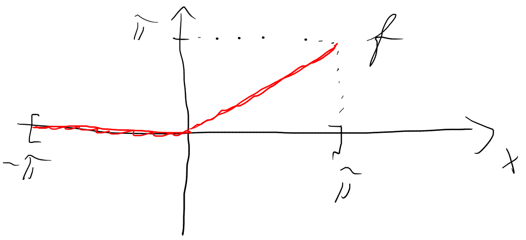
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{f(x^+) + f(x^-)}{2} \quad , x \in (-\pi, \pi)$$

$$f(x^+) = \lim_{t \to x^+} f(t) \quad , \quad f(x^-) = \lim_{t \to x^-} f(t)$$

28.

$$f(x) = \frac{x + |x|}{2} \quad , \quad x \in [-\pi, \pi]$$

$$\leadsto f(x) = \begin{cases} 0 & ; \ x \in [-\pi, 0] \\ x & ; \ x \in (0, \pi] \end{cases}$$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} x\, dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{\pi} x \cos nx\, dx = \dots = \frac{(-1)^n - 1}{\pi n^2}$$

$$= \begin{cases} 0 & , \ n \text{ párne} \\ \frac{-2}{\pi n^2} & , \ n \text{ nepárne} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_0^{\pi} x \sin nx\, dx = \dots = \frac{(-1)^{n-1}}{n} \quad ; \quad n \in \mathbb{N}$$

Fourierov rad: $\displaystyle \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos nx + \frac{(-1)^{n-1}}{n} \sin nx \right]$

$\leadsto f(x)$ je neraslúca a spojitá, preto:

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos nx + \frac{(-1)^{n-1}}{n} \sin nx \right] \quad , x \in (-\pi, \pi)}$$

napr. pre $x = 0$:

$$0 = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \rightsquigarrow \frac{\pi}{4} - \frac{2}{\pi} \cdot \sum_{\ell=1}^{\infty} \frac{1}{(2\ell-1)^2} = 0$$

$$\Rightarrow \sum_{\ell=1}^{\infty} \frac{1}{(2\ell-1)^2} = \frac{\pi^2}{8}$$

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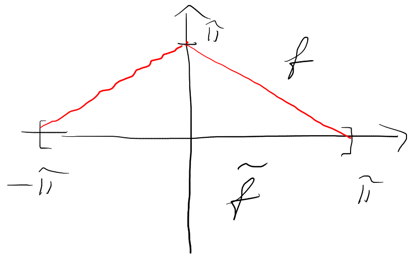
29. $f(x) = \pi - x \; ; \; x \in [0, \pi]$

$\rightsquigarrow$ Fourierov rad pre f? $\longrightarrow$ rozšírenie na $[-\pi, \pi]$

.... párne $\rightsquigarrow$ $\tilde{f}(x) := \begin{cases} \pi - x, & x \in [0, \pi] \\ \pi + x, & x \in [-\pi, 0] \end{cases}$



$\rightsquigarrow b_n = 0, \; n \in \mathbb{N}, \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ — kosínusový rozvoj f

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{f}(x)\,dx = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x)\,dx = \dots = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{f}(x) \cos nx\,dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx\,dx =$$

$$\dots = \frac{2}{\pi n^2}\left[(-1)^{n+1} + 1\right] \quad \begin{cases} 0, & n \text{ párne} \\ \frac{4}{\pi n^2}, & n \text{ nepárne} \end{cases}$$

Fourierov rad: $\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^{n+1} + 1\right] \cos nx = f(x), \; x \in (0, \pi)$

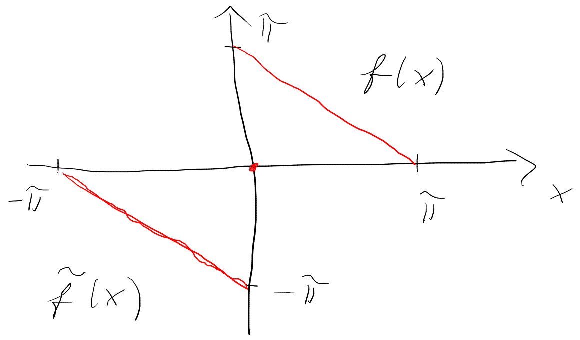
($\tilde{f}$ je spojitá a po častiach monotónna)

30.

$f(x)=\pi-x \quad , \; x\in(0,\pi]$

⇝ sínusový rozvoj pre f $\Rightarrow$ nepárne rozšírenie na $[-\pi,\pi]$:

$$\tilde{f}(x):=\begin{cases} \pi-x & , \; x\in(0,\pi] \\ 0 & , \; x=0 \\ -\pi-x & , \; x\in[-\pi,0) \end{cases}$$



$a_n=0 \; , \; n\in\mathbb{N}_0 \quad , \quad \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\tilde{f}(x)\sin nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}(\pi-x)\sin nx\,dx$$

$$=\ldots=\frac{2}{n}$$

Fourierov rad: $\sum_{n=1}^{\infty}\frac{2}{n}\sin nx = f(x) \; , \; x\in(0,\pi]$

($\tilde{f}$ je po častiach spojitá a monotónna)

⇝ $$\boxed{\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\frac{\pi-x}{2} \; ; \; x\in[0,\pi]}$$