DVOJNÝ INTEGRÁL – OBDĹŽNIK

$f(x,y)\,, \quad (x,y) \in M := [a,b] \times [c,d] \subseteq \mathbb{R}^2$

$$\iint\limits_M f(x,y)\,dx\,dy$$

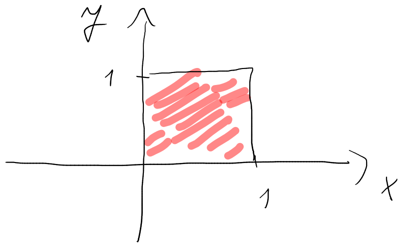
FUBINIHO VETA

ak f je spojitá na M, potom:

$$\iint\limits_M f(x,y)\,dx\,dy = \underbrace{\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx}_{\text{dvojnásobný integrál}}$$

1.

$$I = \iint\limits_M y\,e^{x+y}\,dx\,dy \,, \qquad M = [0,1] \times [0,1]$$



$f(x,y) = y\,e^{x+y}$, spojitá na M $\rightarrow$ Fubini

a) $$I = \int_0^1 \left[\int_0^1 y\,e^{x+y}\,dx\right] dy = \int_0^1 \left[y\,e^{x} \cdot e^{y}\right]_0^1 dy = \int_0^1 \left[y\,e^{y+1} - y\,e^{y}\right] dy =$$

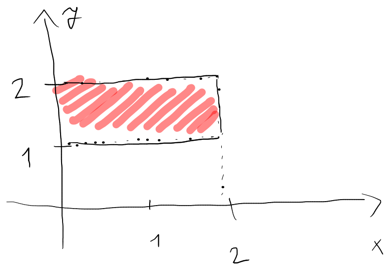
$$= \int_0^1 e^{y}\,y\,(e-1)\,dy = (e-1)\int_0^1 y\,e^{y}\,dy = \begin{vmatrix} u = y \,, & v' = e^{y} \\ u' = 1 \,, & v = e^{y} \end{vmatrix} =$$

$$= (e-1) \cdot \left\{ \left[y\,e^{y}\right]_0^1 - \int_0^1 e^{y}\,dy \right\} = (e-1)\left\{ e - \left[e^{y}\right]_0^1 \right\} = \boxed{e-1}$$

$$I = \int_0^1 \left[\int_0^1 y e^{x+y} \, dy \right] dx = \int_0^1 e^x \left[\int_0^1 y e^y \, dy \right] dx = \int_0^1 e^x \cdot 1 \, dx = \left[e^x \right]_0^1 = \boxed{e-1}$$

2.

$$I = \iint_M \frac{y^2}{1+x^2} \, dx \, dy \quad , \quad M = [0,2] \times [1,2]$$



$f(x,y) = \dfrac{y^2}{1+x^2}$ spojitá na M :

$$I = \int_0^2 \left[\int_1^2 \frac{y^2}{1+x^2} \, dy \right] dx = \int_0^2 \left[\frac{y^3}{3(1+x^2)} \right]_1^2 dx = \int_0^2 \frac{7}{3} \cdot \frac{1}{1+x^2} \, dx =$$

$$= \left[\frac{7}{3} \operatorname{arctg} x \right]_0^2 = \boxed{\frac{7}{3} \operatorname{arctg} 2}$$

3.

$$I = \iint_M x^2 y \cos(x y^2) \, dx \, dy \quad , \quad M = \left[0, \frac{\pi}{2}\right] \times [0,2]$$

$f(x,y) = x^2 y \cos(x y^2)$ spojitá na M :

$$I = \int_0^{\pi/2} \left[\int_0^2 x^2 y \cos(x y^2) \, dy \right] dx = \left| \begin{array}{ll} t = y^2 , & dt = 2y \, dy \\ 0 \rightsquigarrow 0 , & 2 \rightsquigarrow 4 \end{array} \right| =$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^4 \frac{x^2}{2} \cos(xt) \, dt \right] dx = \int_0^{\pi/2} \left[\frac{x}{2} \sin(xt) \right]_0^4 dx = \int_0^{\pi/2} \frac{x}{2} \sin 4x \, dx =$$

$$= \left| \begin{array}{ll} u = \frac{x}{2} , & v' = \sin 4x \\ u' = \frac{1}{2} , & v = -\frac{1}{4} \cos 4x \end{array} \right| = \left[-\frac{x}{8} \cos 4x \right]_0^{\pi/2} + \int_0^{\frac{\pi}{2}} \frac{1}{8} \cos 4x \, dx =$$

$$= \frac{-\pi}{16} + \left[\frac{1}{32}\sin 4x\right]_0^{\frac{\pi}{2}} = \boxed{\frac{-\pi}{16}}$$

$$I = \int_0^2 \left[\int_0^{\pi/2} x^2 y \cos(xy^2)\, dx\right] dy$$

$$\int_0^{\pi/2} x^2 y \cos(xy^2)\, dx = \left| \begin{array}{ll} u = x^2 y, & v' = \cos xy^2 \\ u' = 2xy, & v = \dfrac{\sin xy^2}{y^2} \end{array} \right| =$$

$$= \left[\frac{x}{y}\sin xy^2\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} \frac{2x}{y}\sin xy^2\, dx = \frac{\pi}{2y}\sin\frac{\pi y^2}{2} - \int_0^{\pi/2} \frac{2x}{y}\sin xy^2\, dx =$$

$$= \left| \begin{array}{ll} u = \dfrac{2x}{y}, & v' = \sin xy^2 \\ u' = \dfrac{2}{y}, & v = -\dfrac{\cos xy^2}{y^2} \end{array} \right| = \frac{\pi}{2y}\sin\frac{\pi y^2}{2} - \left\{ \left[-\frac{2x\cos xy^2}{y^3}\right]_0^{\pi/2} + \right.$$

$$\left. + \int_0^{\pi/2} \frac{2}{y^3}\cos xy^2\, dx \right. = \frac{\pi}{2y}\sin\frac{\pi y^2}{2} - \left\{ -\frac{\pi\cos\frac{\pi y^2}{2}}{y^3} + \left[\frac{2}{y^5}\sin xy^2\right]_0^{\pi/2} \right\} =$$

$$= \frac{\pi}{2y}\sin\frac{\pi y^2}{2} + \frac{\pi}{y^3}\cos\frac{\pi y^2}{2} - \frac{2}{y^5}\sin\frac{\pi y^2}{2}$$

$$\Rightarrow I = \int_0^2 \left(\frac{\pi}{2y}\sin\frac{\pi y^2}{2} + \frac{\pi}{y^3}\cos\frac{\pi y^2}{2} - \frac{2}{y^5}\sin\frac{\pi y^3}{2}\right) dy = ???$$

$\rightsquigarrow$ lepšie bolo poradie integrácie najprv y, potom x

$\rightsquigarrow$ z FUBINIHO však potom platí:

$$\int_0^{\pi/2} \left(\frac{\pi}{2y}\sin\frac{\pi y^2}{2} + \frac{\pi}{y^3}\cos\frac{\pi y^2}{2} - \frac{2}{y^5}\sin\frac{\pi y^3}{2}\right) dy = \frac{-\pi}{16}$$

4.

$$I = \iint_M x^y \, dx\, dy \quad , \quad M = [0,1] \times [1,2]$$

$f(x,y)$ je spojitá na M :

$$I = \int_1^2 \left[\int_0^1 x^y dx \right] dy = \int_1^2 \left[\frac{x^{y+1}}{y+1} \right]_0^1 dy = \int_1^2 \frac{1}{y+1} dy =$$

$$= \left[\ln(y+1) \right]_1^2 = \ln 3 - \ln 2 = \boxed{\ln \frac{3}{2}}$$

⇝ opačné poradie integrácie však nevieme klasicky vykonať :

$$I = \int_0^1 \left[\int_1^2 x^y dy \right] dx = \int_0^1 \left[\frac{x^y}{\ln x} \right]_1^2 dx = \int_0^1 \frac{x^2 - x}{\ln x} dx = ???$$

⇝ podľa FUBINIHO ale máme rovnosť :

$$\boxed{\int_0^1 \frac{x^2 - x}{\ln x} dx = \ln \frac{3}{2}}$$

TROJNÝ INTEGRÁL – OBDĹŽNIK

$f(x,y,z)$ spojitá na kvádri $M = [a,b] \times [c,d] \times [g,h]$

FUBINIHO VETA :

$$\iiint_M f(x,y,z)\, dx\, dy\, dz = \underbrace{\int_a^b \left[\int_c^d \left[\int_g^h f(x,y,z)\, dz \right] dy \right] dx}_{\text{trojnásobný integrál}}$$

5.

$$I = \iiint\limits_M x y^2 \sqrt{z}\, dx\, dy\, dz \quad , \quad M = [-2,1] \times [1,3] \times [2,4]$$

$$I = \int_{-2}^{1} \left[\int_{1}^{3} \left[\int_{2}^{4} x y^2 \sqrt{z}\, dz \right] dy \right] dx = \int_{-2}^{1} \left[\int_{1}^{3} \left[x y^2 \cdot \frac{2}{3} z^{3/2} \right]_2^4 dy \right] dx =$$

$$= \int_{-2}^{1} \left[\int_{1}^{3} \frac{16 - 4\sqrt{2}}{3} x y^2\, dy \right] dx = \frac{16 - 4\sqrt{2}}{3} \int_{-2}^{1} \left[\int_{1}^{3} x y^2\, dy \right] dx =$$

$$= \frac{16 - 4\sqrt{2}}{3} \cdot \int_{-2}^{1} \left[\frac{x}{3} y^3 \right]_1^3 dx = \frac{16 - 4\sqrt{2}}{3} \cdot \int_{-2}^{1} \frac{26}{3} \cdot x\, dx =$$

$$= \frac{26 \cdot 4 \cdot (4 - \sqrt{2})}{9} \cdot \int_{-2}^{1} x\, dx = \frac{26 \cdot 4 \cdot (4 - \sqrt{2})}{9} \cdot \left[\frac{x^2}{2} \right]_{-2}^{1} =$$

$$= \frac{26 \cdot 4 \cdot (4 - \sqrt{2})}{9} \cdot \frac{(-3)}{2} = \boxed{\frac{52 \cdot (\sqrt{2} - 4)}{3}}$$

6.

$$I = \iiint\limits_M 2 e^{3x + 2y + z}\, dx\, dy\, dz \quad , \quad M = [0,1]^3$$

$f(x,y,z) = 2 e^{3x+2y+z}$ spojitá na M :

$$I = \int_0^1 \left[\int_0^1 \left[\int_0^1 2 e^{3x+2y+z}\, dy \right] dx \right] dz =$$

$$= \int_0^1 \left[\int_0^1 \left[e^{3x+2y+z} \right]_0^1 dx \right] dz = \int_0^1 \left[\int_0^1 e^{3x+z} (e^2 - 1)\, dx \right] dz =$$

$$= (e^2 - 1) \cdot \int_0^1 \left[\frac{1}{3} e^{3x+z} \right]_0^1 dz = (e^2 - 1) \int_0^1 \frac{e^3 - 1}{3} e^z\, dz = \boxed{\frac{(e^2-1)(e^3-1)(e-1)}{3}}$$