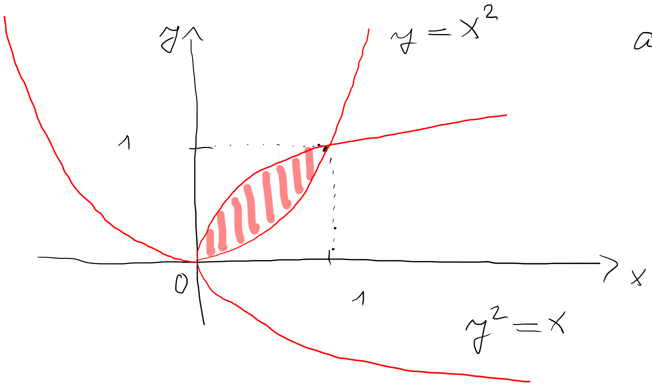
# DVOJNÝ INTEGRÁL – ELEMENTÁRNE OBLASTI

7.

$$I = \iint_M (x^2 + y)\,dx\,dy \;, \quad M: \; y = x^2 \;, \; y^2 = x$$




a) M: $0 \le x \le 1$

$x^2 \le y \le \sqrt{x}$

elementárna oblasť vzhľadom na os x

$$I = \int_0^1 \left[ \int_{x^2}^{\sqrt{x}} (x^2+y)\,dy \right] dx = \int_0^1 \left[ yx^2 + \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx =$$

$$= \int_0^1 \left( x^{5/2} + \frac{x}{2} - \frac{3}{2}x^4 \right) dx = \left[ \frac{2}{7} x^{7/2} + \frac{x^2}{4} - \frac{3}{10} x^5 \right]_0^1 = \boxed{\frac{33}{140}}$$

b) M: $0 \le y \le 1$ ; $y^2 \le x \le \sqrt{y}$

elementárna oblasť vzhľadom na os y

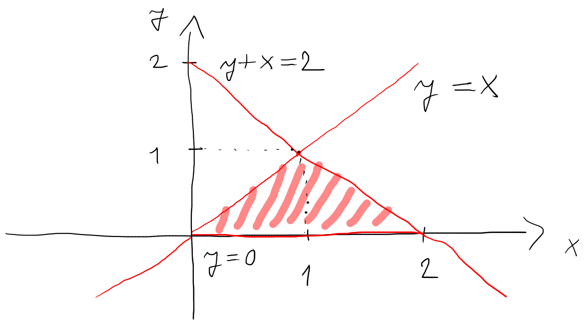
$$I = \int_0^1 \left[ \int_{y^2}^{\sqrt{y}} (x^2+y)\,dx \right] dy = \int_0^1 \left[ \frac{x^3}{3} + yx \right]_{y^2}^{\sqrt{y}} dy =$$

$$= \int_0^1 \left( \frac{4}{3} y^{3/2} - \frac{y^6}{3} - y^3 \right) dy = \left[ \frac{8}{15} y^{5/2} - \frac{y^7}{21} - \frac{y^4}{4} \right]_0^1 =$$

$$= \boxed{\frac{33}{140}}$$

8.

$$I=\iint_M (x-y)\,dx\,dy \quad ; \quad M: \; y=0, \; y=x, \; y+x=2$$




$$M: \quad \left.\begin{array}{l} 0 \le y \le 1 \\ y \le x \le 2-y \end{array}\right\} \text{elementárna vzhľadom na os } y$$

$$I=\int_0^1\left[\int_y^{2-y}(x-y)\,dx\right]dy=\int_0^1\left[\frac{x^2}{2}-xy\right]_y^{2-y}dy=$$

$$=\int_0^1\left(\frac{(2-y)^2}{2}-(2-y)y-\frac{y^2}{2}+y^2\right)dy=\int_0^1\left(2y^2-4y+2\right)dy=$$

$$=\left[\frac{2}{3}y^3-2y^2+2y\right]_0^1=\boxed{\frac{2}{3}}$$

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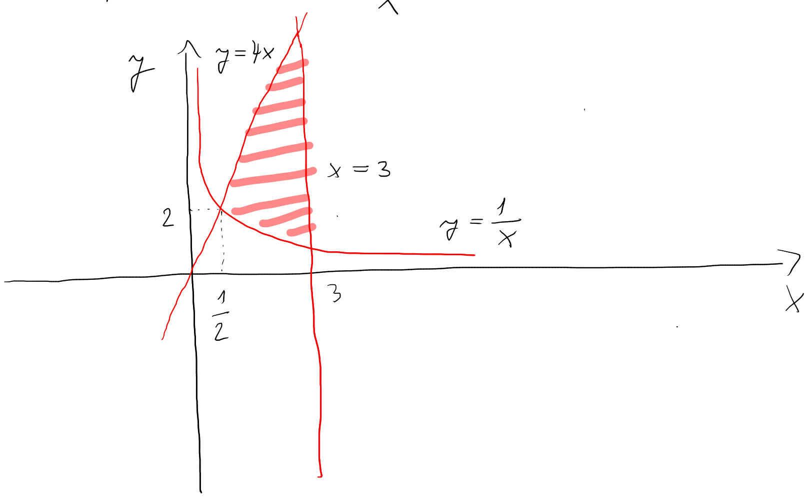
9.

$$I=\iint_M \frac{x^2}{y^2} \quad , \quad M: \; y=\frac{1}{x}, \; y=4x, \; x=3$$




$M:$

$$\frac{1}{2} \le x \le 3$$

$$\frac{1}{x} \le y \le 4x$$

$$I = \int_{1/2}^{3} \left[ \int_{\frac{1}{x}}^{4x} \frac{x^2}{y^2}\, dy \right] dx = \int_{\frac{1}{2}}^{3} \left[ -\frac{x^2}{y} \right]_{\frac{1}{x}}^{4x} dx = \int_{1/2}^{3} \left( x^3 - \frac{x}{4} \right) dx =$$

$$= \left[ \frac{x^4}{4} - \frac{x^2}{8} \right]_{1/2}^{3} = \frac{81}{4} - \frac{9}{8} - \frac{1}{64} + \frac{1}{32} = \boxed{\frac{1225}{64}}$$

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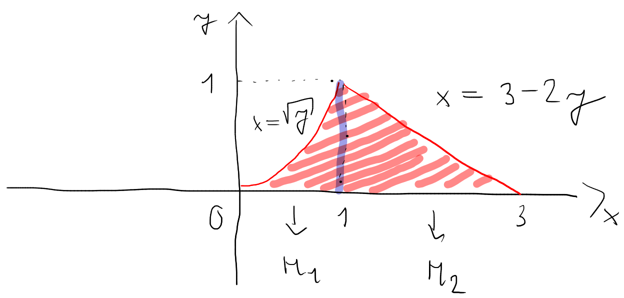
## DVOJNÝ INTEGRÁL – ZÁMENA INTEGRÁCIE

10. $$I = \int_0^1 \left[ \int_{\sqrt{y}}^{3-2y} f(x,y)\, dx \right] dy$$ ⇝ zameniť poradie integ.




$M:$ $0 \le y \le 1$

$\sqrt{y} \le x \le 3-2y$

$M = M_1 \cup M_2$

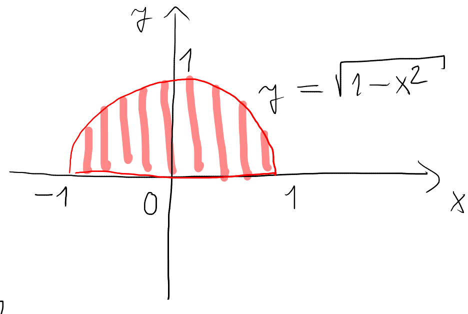
$M_1:$ $0 \le x \le 1$, $0 \le y \le x^2$

$M_2:$ $1 \le x \le 3$, $0 \le y \le \frac{3-x}{2}$

$$\boxed{I = \int_0^1 \left[ \int_0^{x^2} f(x,y)\, dy \right] dx + \int_1^3 \left[ \int_0^{\frac{3-x}{2}} f(x,y)\, dy \right] dx}$$

11.

$$I = \int_{-1}^{1} \left[ \int_{0}^{\sqrt{1-x^2}} f(x,y)\, dy \right] dx$$




$M: \quad -1 \le x \le 1 \quad , \quad 0 \le y \le \sqrt{1-x^2}$

$M: \quad 0 \le y \le 1 \quad , \quad -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

$$\boxed{I = \int_{0}^{1} \left[ \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\, dx \right] dy}$$

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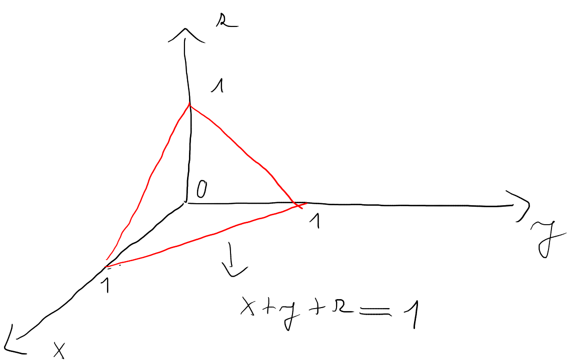
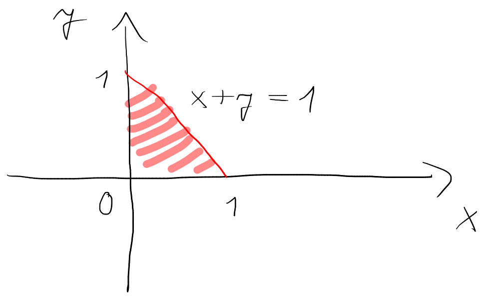
## TROJNÝ INTEGRÁL – ELEMENTÁRNE OBLASTI

12.

$$I = \iiint_M \frac{1}{x+y+1}\, dx\,dy\,dz \quad , \quad M: \; x \ge 0 \, , \; y \ge 0 \, , \; z \ge 0$$

$$x + y + z \le 1$$




$\underline{M:} \quad 0 \le x \le 1 \quad , \quad 0 \le y \le 1-x \quad , \quad 0 \le z \le 1-x-y$

$$I = \int_0^1 \left[ \int_0^{1-x} \left[ \int_0^{1-x-y} \frac{1}{x+y+1} \, dz \right] dy \right] dx =$$

$$= \int_0^1 \left[ \int_0^{1-x} \left[ \frac{z}{x+y+1} \right]_0^{1-x-y} dy \right] dx = \int_0^1 \left[ \int_0^{1-x} \left( \frac{1-x-y}{x+y+1} \right) dy \right] dx$$

$$= \int_0^1 \left[ \int_0^{1-x} \left( \frac{2}{x+y+1} - 1 \right) dy \right] dx = \int_0^1 \Big[ 2\ln(x+y+1) - y \Big]_0^{1-x} dx =$$

$$= \int_0^1 \Big( 2\ln 2 - 1 + x - 2\ln(x+1) \Big) dx = \left[ (2\ln 2 - 1)x + \frac{x^2}{2} \right]_0^1 -$$

$$2\int_0^1 \ln(x+1)\, dx = 2\ln 2 - \frac{1}{2} - 2\int_0^1 \ln(x+1) dx = \begin{vmatrix} u' = 1 & , \; v = \ln(x+1) \\ u = x & , \; v' = \frac{1}{x+1} \end{vmatrix}$$

$$= \ln 4 - \frac{1}{2} - 2\left\{ \Big[ x\ln(x+1) \Big]_0^1 - \int_0^1 \frac{x}{x+1} dx \right\} = \ln 4 - \frac{1}{2} - 2\ln 2 + 2\int_0^1 \left( 1 - \frac{1}{x+1} \right) dx$$

$$= -\frac{1}{2} + 2\Big[ x - \ln(x+1) \Big]_0^1 = -\frac{1}{2} + 2(1 - \ln 2) = \boxed{\frac{3}{2} - \ln 4}$$

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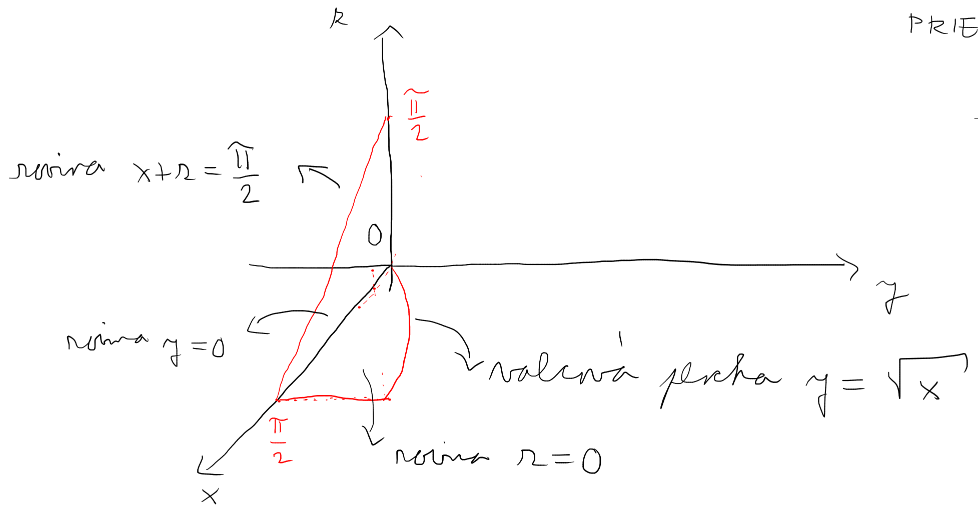
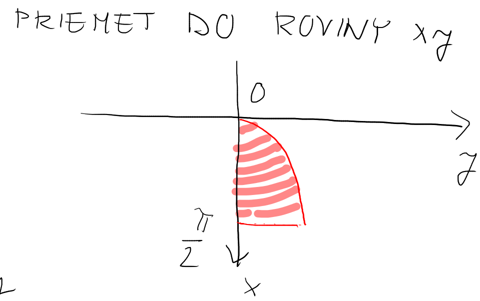
13.

$$I = \iiint_M y \cos(x+z)\, dx\, dy\, dz \quad , \quad M: \; y = \sqrt{x} \; , \; y = 0$$
$$z = 0 \; , \; x + z = \frac{\pi}{2}$$




$\underline{M:}\quad 0 \le x \le \frac{\pi}{2}\ ,\quad 0 \le y \le \sqrt{x}\ ;\quad 0 \le z \le \frac{\pi}{2} - x$

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$$I = \int_0^{\frac{\pi}{2}} \left[ \int_0^{\sqrt{x}} \left[ \int_0^{\frac{\pi}{2}-x} y \cos(x+z)\, dz \right] dy \right] dx =$$

$$= \int_0^{\pi/2} \left[ \int_0^{\sqrt{x}} \left[ y \sin(x+z) \right]_0^{\frac{\pi}{2}-x} dy \right] dx = \int_0^{\pi/2} \left[ \int_0^{\sqrt{x}} (y - y \sin x)\, dy \right] dx$$

$$= \int_0^{\pi/2} \left[ \frac{y^2}{2} (1 - \sin x) \right]_0^{\sqrt{x}} dx = \int_0^{\pi/2} \frac{x}{2} (1 - \sin x)\, dx =$$

$$= \left| \begin{array}{ll} u = \frac{x}{2}\ , & v' = 1 - \sin x \\ u' = \frac{1}{2}\ , & v = x + \cos x \end{array} \right| = \left[ \frac{x}{2} (x + \cos x) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{x + \cos x}{2}\, dx =$$

$$= \frac{\pi^2}{8} - \left[ \frac{x^2}{4} + \frac{1}{2} \sin x \right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{8} - \frac{\pi^2}{16} - \frac{1}{2} = \boxed{\frac{\pi^2 - 8}{16}}$$