TRANSFORMÁCIE DVOJNÝCH INTEGRÁLOV

$$\underline{\text{oblasť } M \subseteq \mathbb{R}^2} \xleftarrow{(g,h)} \underline{\text{oblasť } N \subseteq \mathbb{R}^2}$$

(x,y) $\qquad$ (u,v)

$x = g(u,v)$, $y = h(u,v)$; g, h majú spojité deriv.

Jacobiho matica zobrazenia (g,h) : $\begin{pmatrix} g'_u & g'_v \\ h'_u & h'_v \end{pmatrix}$

jacobián zobrazenia (g,h) : $J(u,v) = \det\begin{pmatrix} g'_u , g'_v \\ h'_u , h'_v \end{pmatrix} \neq 0$

$$\iint_M f(x,y)\,dx\,dy = \iint_N f(g(u,v), h(u,v)) \cdot |J(u,v)|\,du\,dv$$

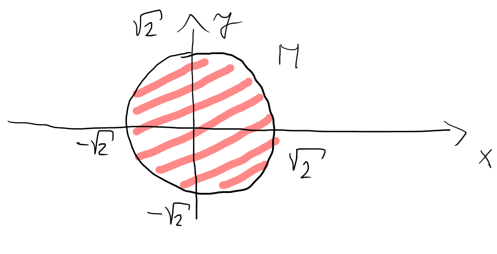
$\rightsquigarrow$ transformácia do polárnych súradníc :

$x = \rho\cos\varphi$, $y = \rho\sin\varphi$; $\rho \in (0,\infty)$, $\varphi \in (0, 2\pi)$

$J(\rho,\varphi) = \det\begin{pmatrix} x'_\rho & y'_\rho \\ x'_\varphi & y'_\varphi \end{pmatrix} = \rho$

14.

$$I = \iint_M (1 - 2x - 3y)\,dx\,dy \quad , M : x^2 + y^2 \le 2$$



⇝ polárne súradnice : $x=\rho\cos\varphi$, $y=\rho\sin\varphi$

M : $0\le\varphi\le 2\pi$; $0\le\rho\le\sqrt{2}$

$$I=\int_0^{2\pi}\left[\int_0^{\sqrt{2}}\left[1-2\rho\cos\varphi-3\rho\sin\varphi\right]\rho\cdot d\rho\right]d\varphi=$$

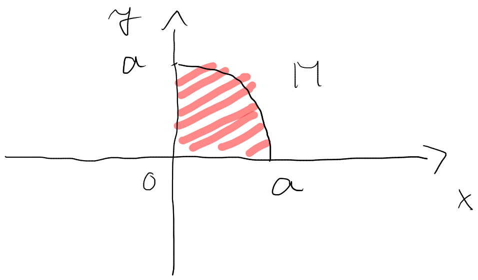
$$=\int_0^{2\pi}\left[\int_0^{\sqrt{2}}\left(\rho-2\rho^2\cos\varphi-3\rho^2\sin\varphi\right)d\rho\right]d\varphi=$$

$$=\int_0^{2\pi}\left[\frac{\rho^2}{2}-\frac{2}{3}\rho^3\cos\varphi-\rho^3\sin\varphi\right]_0^{\sqrt{2}}d\varphi=\int_0^{2\pi}\left(1-\frac{4\sqrt{2}}{3}\cos\varphi-2\sqrt{2}\sin\varphi\right)d\varphi$$

$$=\left[\varphi-\frac{4\sqrt{2}}{3}\sin\varphi+2\sqrt{2}\cos\varphi\right]_0^{2\pi}=2\pi$$

15.

$$I=\iint_M (x^2+y^2)\,dx\,dy\ ,\quad M:\ x^2+y^2\le a^2\ ,\ x\ge 0,\ y\ge 0\ ,\ a>0$$



⇝ polárne súradnice : $x=\rho\cos\varphi$, $y=\rho\sin\varphi$

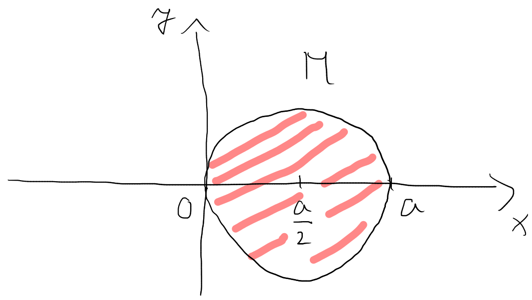
M : $0\le\varphi\le\frac{\pi}{2}$, $0\le\rho\le a$

$$I=\int_0^{\pi/2}\left[\int_0^{a}\left[\rho^2\cos^2\varphi+\rho^2\sin^2\varphi\right]\rho\,d\rho\right]d\varphi=\int_0^{\pi/2}\left[\int_0^{a}\rho^3\,d\rho\right]d\varphi=$$

$$=\int_0^{\pi/2}\left[\frac{1}{4}\rho^4\right]_0^a d\varphi=\int_0^{\pi/2}\frac{a^4}{4}\,d\varphi=\frac{a^4}{4}\left[\varphi\right]_0^{\pi/2}=\frac{\pi a^4}{8}$$

16.

$$I = \iint_M \sqrt{a^2 - x^2 - y^2}\, dx\, dy \quad ; \quad M: x^2+y^2 \le ax \quad , a>0$$



$$M: \left(x-\frac{a}{2}\right)^2 + y^2 \le \left(\frac{a}{2}\right)^2$$

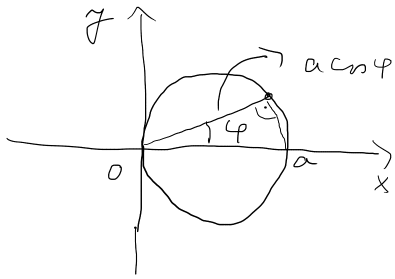
$\leadsto$ polárne súradnice :

$$x = \rho\cos\varphi \quad , \quad y = \rho\sin\varphi$$

$$x^2+y^2 \le a\cdot x \leadsto \rho^2\cos^2\varphi + \rho^2\sin^2\varphi \le a\rho\cos\varphi \leadsto \rho^2 \le a\rho\cos\varphi$$

$$\leadsto \rho \le a\cos\varphi$$

$$M: \quad -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \quad ; \quad 0 \le \rho \le a\cos\varphi$$



$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_0^{a\cos\varphi} \left[\sqrt{a^2 - \rho^2\cos^2\varphi - \rho^2\sin^2\varphi}\; \rho\, d\rho \right] \right] d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \left[\int_0^{a\cos\varphi} \sqrt{a^2-\rho^2}\; \rho\, d\rho \right] d\varphi = \left| \begin{matrix} t = a^2 - \rho^2 & , & dt = -2\rho\, d\rho \\ 0 \leadsto a^2 & , & a\cos\varphi \leadsto a^2\sin^2\varphi \end{matrix} \right|$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_{a^2}^{a^2\sin^2\varphi} -\frac{1}{2}\sqrt{t}\, dt \right] d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\frac{1}{3} t^{3/2} \right]_{a^2}^{a^2\sin^2\varphi} d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-\frac{1}{3} a^3 |\sin\varphi|^3 + \frac{1}{3} a^3 \right) d\varphi = \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - |\sin\varphi|^3 \right) d\varphi =$$

$$= \frac{a^3}{3} \cdot 2 \int_0^{\pi/2} (1-\sin^3\varphi)\,d\varphi = \frac{2a^3}{3}\cdot[\varphi]_0^{\frac{\pi}{2}} - \frac{2a^3}{3}\int_0^{\frac{\pi}{2}} \sin^3\varphi\,d\varphi$$

$$= \frac{\pi a^3}{3} - \frac{2a^3}{3}\int_0^{\frac{\pi}{2}} \underbrace{\sin^3\varphi\,d\varphi}_{\substack{\sin^2\varphi\cdot\sin\varphi\,d\varphi \\ =(1-\cos^2\varphi)\sin\varphi\,d\varphi}} = \left| \begin{array}{ll} s=\cos\varphi, & ds = -\sin\varphi\,d\varphi \\ 0\to 1, & \frac{\pi}{2}\to 0\end{array}\right|$$

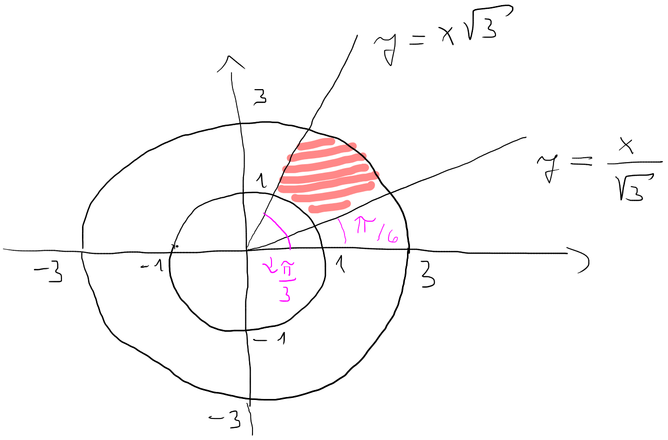
$$= \frac{\pi a^3}{3} - \frac{2a^3}{3}\int_1^0 (1-s^2)(-ds) = \frac{\pi a^3}{3} - \frac{2a^3}{3}\int_0^1(1-s^2)\,ds =$$

$$= \frac{\pi a^3}{3} - \frac{2a^3}{3}\left[s-\frac{s^3}{3}\right]_0^1 = \frac{\pi a^3}{3} - \frac{2a^3}{3}\cdot\frac{2}{3} = \frac{(3\pi-4)a^3}{9}$$

//

17.

$$I = \iint_M \operatorname{arctg}\frac{y}{x}\,dx\,dy \quad ; M: 1\le x^2+y^2\le 9,\ y\le x\sqrt{3},\ y\ge \frac{x}{\sqrt{3}}$$



polárne súradnice:

$x = \rho\cos\varphi, \quad y = \rho\sin\varphi$

$M: \quad \frac{\pi}{6} \le \varphi \le \frac{\pi}{3},$

$1 \le \rho \le 3$

$$I = \int_{\pi/6}^{\pi/3}\left[\int_1^3 \operatorname{arctg}\left(\frac{\rho\sin\varphi}{\rho\cos\varphi}\right)\rho\,d\rho\right]d\varphi = \int_{\pi/6}^{\pi/3}\left[\int_1^3 \operatorname{arctg}(\operatorname{tg}\varphi)\,\rho\,d\rho\right]d\varphi$$

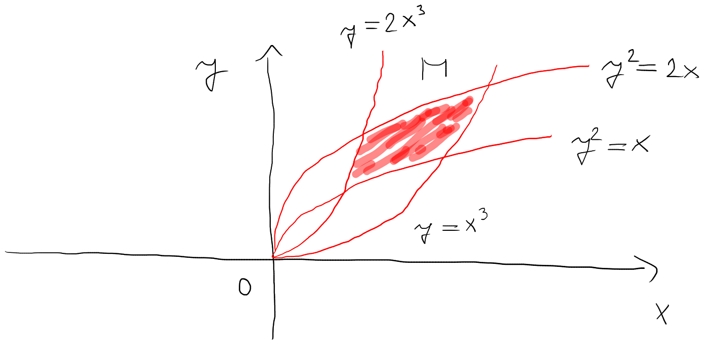
$$= \int_{\pi/6}^{\pi/3} \left[\int_1^3 \varphi \cdot \rho \, d\rho \right] d\varphi = \left(\int_{\pi/6}^{\pi/3} \varphi \, d\varphi \right) \cdot \left(\int_1^3 \rho \, d\rho \right) =$$

$$= \left[\frac{\varphi^2}{3} \right]_{\pi/6}^{\pi/3} \cdot \left[\frac{\rho^2}{2} \right]_1^3 = \frac{\pi^2}{3}\left(\frac{1}{9} - \frac{1}{36} \right) \cdot 4 = \frac{\pi^2}{12} \cdot \frac{4-1}{36} =$$

$$= \frac{\pi^2}{144}$$

18.

$$I = \iint_M xy \, dx \, dy \; ; \quad M: \; y = 2x^3 \, , \; y = x^3 \, , \; y^2 = x \, , \; y^2 = 2x$$



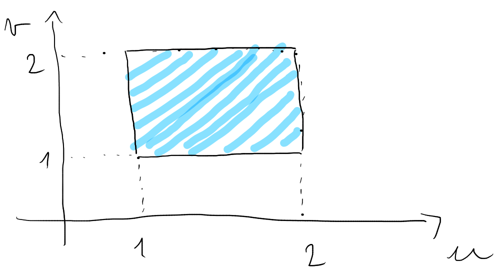
→ transformácia $(x,y) \to (u,v)$

$$y = u x^3 \, , \quad y^2 = v x$$

$$\Rightarrow \quad x = \left(\frac{v}{u^2} \right)^{1/5} , \quad y = \left(\frac{v^3}{u} \right)^{1/5}$$

→ medze pre u, v :

$$u = \frac{y}{x^3} \, , \quad v = \frac{y^2}{x} \quad \Rightarrow \quad M: \; 1 \le u \le 2 \, , \quad 1 \le v \le 2$$



→ jacobián : $x'_u = -\frac{2}{5} \cdot \left(\frac{v}{u^7} \right)^{1/5} , \quad x'_v = \frac{1}{5} \cdot \frac{1}{(v^4 u^2)^{1/5}}$

$$y'_u = -\frac{1}{5}\left(\frac{v^3}{u^6}\right)^{1/5}, \quad y'_v = \frac{3}{5}\cdot\frac{1}{(v^2u)^{1/5}}$$

$$J(u,v) = \det\begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \end{pmatrix} = -\frac{6}{25}\frac{1}{(vu^8)^{1/5}} + \frac{1}{25}\cdot\frac{1}{(vu^8)^{1/5}}$$

$$= -\frac{1}{5}\frac{1}{(vu^8)^{1/5}}$$

$$I = \int_1^2\left[\int_1^2 \left(\frac{v}{u^2}\right)^{1/5}\cdot\left(\frac{v^3}{u}\right)^{1/5}\cdot\left|-\frac{1}{5}\cdot\frac{1}{(vu^8)^{1/5}}\right| du\right]dv =$$

$$= \frac{1}{5}\int_1^2\left[\int_1^2 \left(\frac{v^3}{u^{11}}\right)^{1/5} du\right]dv = \frac{1}{5}\left[\int_1^2 v^{3/5}\,dv\right]\cdot\left[\int_1^2 u^{-\frac{11}{5}}\,du\right] =$$

$$= \frac{1}{5}\left[\frac{5}{8}v^{8/5}\right]_1^2\cdot\left[-\frac{5}{6}v^{-\frac{6}{5}}\right]_1^2 = \frac{5}{48}\cdot\left(2^{8/5}-1\right)\left(1-2^{-6/5}\right)$$

TRANSFORMÁCIE TROJNÝCH INTEGRÁLOV

→ valcové súradnice: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $z = z$

$$J(\rho,\varphi,z) = \rho$$

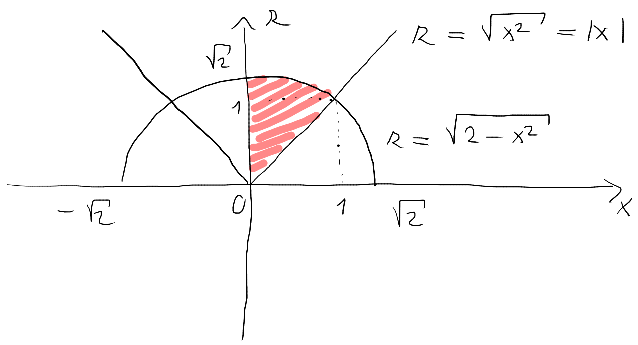
→ sférické súradnice: $x = \rho\cos\varphi\sin\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\theta$

$$J(\rho,\varphi,\theta) = -\rho^2\sin\theta$$

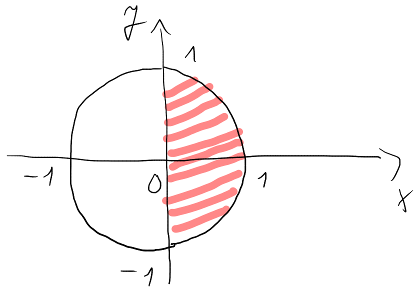
19.

$$I = \iiint_M xz\,dx\,dy\,dz \quad ; \quad M:\ z=\sqrt{2-x^2-y^2}\,,\ z=\sqrt{x^2+y^2}\,,\ x=0$$
$$x \geq 0$$

→ priemet do roviny xz :



→ priemet do roviny xy :



valcové súradnice : $x=\rho\cos\varphi\,,\ y=\rho\sin\varphi\,,\ z=z$

$M:\ -\frac{\pi}{2}\leq\varphi\leq\frac{\pi}{2}\,,\quad 0\leq\rho\leq 1\ ;\quad \rho\leq z\leq\sqrt{2-\rho^2}$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^1\left[\int_{\rho}^{\sqrt{2-\rho^2}}(\rho\cos\varphi)\cdot z\cdot\rho\,dz\right]d\rho\right]d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^1\left[\int_{\rho}^{\sqrt{2-\rho^2}}\rho^2\cos\varphi\cdot z\,dz\right]d\rho\right]d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^1\left[\rho^2\cos\varphi\cdot\frac{z^2}{2}\right]_{\rho}^{\sqrt{2-\rho^2}}d\rho\right]d\varphi$$

$$= \int_{-\pi/2}^{\pi/2} \left[\int_0^1 \left(\varrho^2(2-\varrho^2) - \varrho^2\cdot\varrho^2\right)\frac{\cos\varphi}{2}\, d\varrho\right] d\varphi =$$

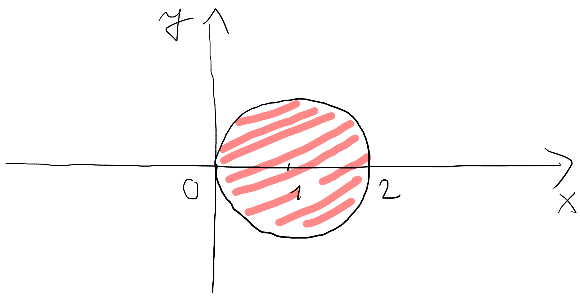
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^1 (\varrho^2-\varrho^4)\cos\varphi\, d\varrho\right] d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\varrho^3}{3} - \frac{\varrho^5}{5}\right]_0^1 \cos\varphi\, d\varphi =$$

$$= \int_{-\pi/2}^{\pi/2} \frac{2}{15}\cos\varphi\, d\varphi = \frac{2}{15}\left[\sin\varphi\right]_{-\pi/2}^{\pi/2} = \frac{4}{15}$$

20.

$$I = \iiint_M z\sqrt{x^2+y^2}\,dx\,dy\,dz \quad ; \quad M: \ x^2+y^2=2x, \ z=0, \ z=a, \quad a>0$$

→ průmět do roviny xy :



→ válcové souřadnice :

$$x = \varrho\cos\varphi, \quad y = \varrho\sin\varphi, \quad z = z$$

$$M: \ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \ ; \quad 0 \le \varrho \le 2\cos\varphi \ ; \quad 0 \le z \le a$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2\cos\varphi}\left[\int_0^a z\cdot\varrho\cdot\varrho\, dz\right] d\varrho\right] d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2\cos\varphi}\left[\varrho^2\frac{z^2}{2}\right]_0^a d\varrho\right] d\varphi =$$

$$= \int_{-\pi/2}^{\pi/2} \left[\int_0^{2\cos\varphi} \frac{a^2}{2} \rho^2 \, d\rho \right] d\varphi = \frac{a^2}{2} \int_{-\pi/2}^{\pi/2} \left[\rho^3/3 \right]_0^{2\cos\varphi} d\varphi =$$

$$= \frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{8}{3} \cos^3\varphi \right) d\varphi = \frac{4}{3} a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^3\varphi \, d\varphi}_{(1-\sin^2\varphi)\cos\varphi \, d\varphi} =$$

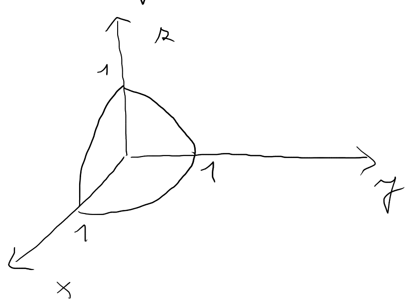
$$= \left| \begin{array}{ll} t = \sin\varphi & , dt = \cos\varphi \, d\varphi \\ -\frac{\pi}{2} \rightsquigarrow -1 & , \frac{\pi}{2} \rightsquigarrow 1 \end{array} \right| = \frac{4}{3} a^2 \cdot \int_{-1}^{1} (1 - t^2) \, dt =$$

$$= \frac{4}{3} a^2 \left[t - \frac{t^3}{3} \right]_{-1}^{1} = \frac{4}{3} a^2 \cdot \frac{4}{3} = \frac{16 a^2}{9}$$

21.

$$I = \iiint_M \sqrt{x^2 + y^2 + z^2} \, dx\,dy\,dz \; ; \quad M: x^2 + y^2 + z^2 \le 1 \; , \; x, y, z \ge 0$$

$\rightsquigarrow$ M je časť jednotkovej gule v I. oktante :



$\rightsquigarrow$ sférické súradnice : $x = \rho \cos\varphi \sin\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\theta$

$M: \quad 0 \le \varphi \le \frac{\pi}{2} \; ; \quad 0 \le \theta \le \frac{\pi}{2} \; ; \; 0 \le \rho \le 1$

$$I=\int_0^{\pi/2}\left[\int_0^{\pi/2}\left[\int_0^1 \sqrt{\varrho^2\cos^2\varphi\sin^2\theta+\varrho^2\sin^2\varphi\sin^2\theta+\varrho^2\cos^2\theta}\;\varrho^2\sin\theta\,d\varrho\right]d\theta\right]d\varphi$$

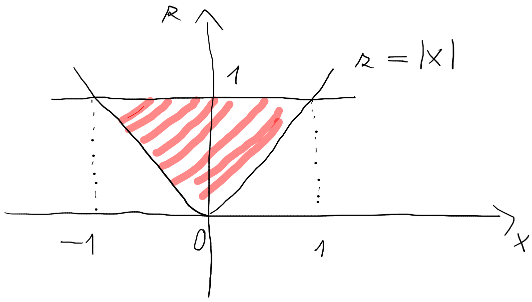
$$=\int_0^{\pi/2}\left[\int_0^{\pi/2}\left[\int_0^1 \varrho^3\sin\theta\,d\varrho\right]d\theta\right]d\varphi=\int_0^{\pi/2}\left[\int_0^{\pi/2}\left[\frac{\varrho^4}{4}\sin\theta\right]_0^1 d\theta\right]d\varphi=$$

$$=\int_0^{\pi/2}\left[\int_0^{\pi/2}\frac{1}{4}\sin\theta\,d\theta\right]d\varphi=\int_0^{\frac{\pi}{2}}\left[-\frac{1}{4}\cos\theta\right]_0^{\pi/2}d\varphi=\int_0^{\pi/2}\frac{1}{4}\,d\varphi=\frac{1}{4}\left[\varphi\right]_0^{\frac{\pi}{2}}=\frac{\pi}{8}$$

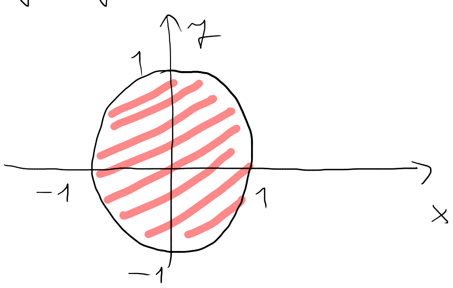
22.

$$I=\iiint_M z\,dx\,dy\,dz\quad,\quad M:\ z=\sqrt{x^2+y^2}\quad,\quad z=1$$

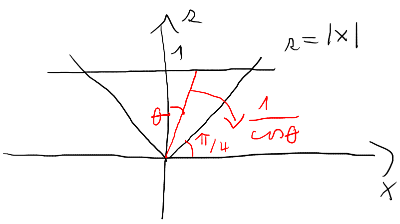
→ priemet do roviny xz :



→ priemet do roviny xy :



→ sférické súradnice : $x=\varrho\cos\varphi\sin\theta\ ,\ y=\varrho\sin\varphi\sin\theta\ ,\ z=\varrho\cos\theta$



$$M:\ 0\le\varphi\le 2\pi\ ;\ 0\le\theta\le\pi/4$$

$$0\le\varrho\le\frac{1}{\cos\theta}$$

$$I = \int_0^{2\pi} \left[\int_0^{\pi/4} \left[\int_0^{\frac{1}{\cos\theta}} \rho\cos\theta \cdot \rho^2 \sin\theta \, d\rho \right] d\theta \right] d\varphi =$$

$$= \int_0^{2\pi} \left[\int_0^{\pi/4} \left[\int_0^{1/\cos\theta} \rho^3 \cos\theta \sin\theta \, d\rho \right] d\theta \right] d\varphi =$$

$$= \int_0^{2\pi} \left[\int_0^{\pi/4} \left[\frac{\rho^4}{4} \cos\theta \sin\theta \right]_0^{\frac{1}{\cos\theta}} d\theta \right] d\varphi = \int_0^{2\pi} \left[\int_0^{\pi/4} \frac{\sin\theta}{4\cos^3\theta} \, d\theta \right] d\varphi =$$

$$= \left| \begin{array}{ll} t = \cos\theta & , \; dt = -\sin\theta \, d\theta \\ 0 \rightsquigarrow 1 & , \; \frac{\pi}{4} \rightsquigarrow \frac{1}{\sqrt{2}} \end{array} \right| = \int_0^{2\pi} \left[\int_1^{1/\sqrt{2}} \frac{-dt}{4t^3} \right] d\varphi =$$

$$= \int_0^{2\pi} \left[\frac{1}{8t^2} \right]_1^{1/\sqrt{2}} d\varphi = \int_0^{2\pi} \frac{1}{8} \, d\varphi = \frac{1}{8} \left[\varphi \right]_0^{2\pi} = \frac{\pi}{4}$$