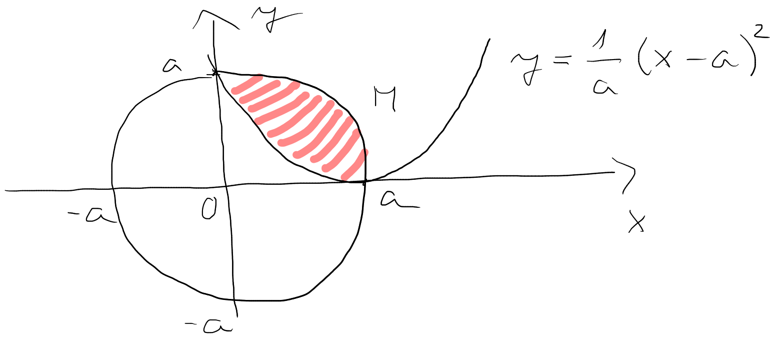
APLIKÁCIE DVOJNÝCH INTEGRÁLOV

(23.)

obsah rovinnej oblasti M : $y = \frac{1}{a}(x-a)^2$, $x^2+y^2=a^2$, $a>0$

$$S(M) = \iint_M dx\,dy$$



M: $0 \le x \le a$

$\frac{1}{a}(x-a)^2 \le y \le \sqrt{a^2-x^2}$

$$S(M) = \int_0^a \left[\int_{\frac{(x-a)^2}{a}}^{\sqrt{a^2-x^2}} dy \right] dx = \int_0^a \left[y \right]_{\frac{(x-a)^2}{a}}^{\sqrt{a^2-x^2}} dx = \int_0^a \left(\sqrt{a^2-x^2} - \frac{1}{a}(x-a)^2 \right) dx$$

$$= \int_0^a \sqrt{a^2-x^2}\, dx - \frac{1}{a}\left[\frac{(x-a)^3}{3} \right]_0^a = \int_0^a \sqrt{a^2-x^2}\, dx - \frac{a^2}{3} =$$

$$= \left| \begin{array}{ll} x = a\sin t \ , & dx = a\cos t\, dt \\ 0 \rightsquigarrow 0 \ , & a \rightsquigarrow \frac{\pi}{2} \end{array} \right| = -\frac{a^2}{3} + \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2 t}\; a\cos t\, dt$$

$$= -\frac{a^2}{3} + a^2 \int_0^{\pi/2} |\cos t| \cos t\, dt = -\frac{a^2}{3} + a^2 \cdot \int_0^{\pi/2} \cos^2 t\, dt =$$

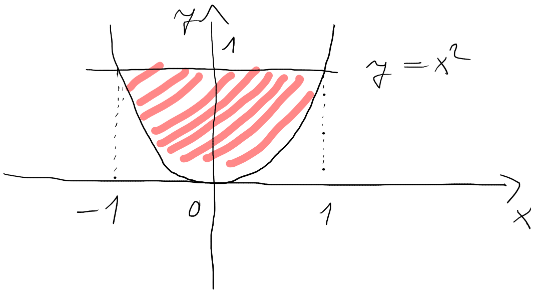
$$= -\frac{a^2}{3} + a^2 \int_0^{\pi/2} \frac{1+\cos 2t}{2}\, dt = -\frac{a^2}{3} + a^2 \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{\pi/2} =$$

$$= -\frac{a^2}{3} + \frac{a^2\pi}{4} = \frac{(3\pi-4)a^2}{12}$$

24.

objem oblasti pod grafom f-cie $f(x,y)\geq 0$ na množine $M\subseteq \mathbb{R}^2$

$f(x,y) = x^2+y^2$; $M: \; y=x^2, \; y=1$



$M: \; -1\leq x\leq 1, \quad x^2\leq y\leq 1$

$$V = \iint_M f(x,y)\,dx\,dy = \iint_M (x^2+y^2)\,dx\,dy = \int_{-1}^{1}\left[\int_{x^2}^{1}(x^2+y^2)\,dy\right]dx$$

$$= \int_{-1}^{1}\left[x^2y+\frac{y^3}{3}\right]_{x^2}^{1}dx = \int_{-1}^{1}\left(x^2+\frac{1}{3}-x^4-\frac{x^6}{3}\right)dx =$$

$$= \left[\frac{x^3}{3}+\frac{x}{3}-\frac{x^5}{5}-\frac{x^7}{21}\right]_{-1}^{1} = 2\cdot\left(\frac{1}{3}+\frac{1}{3}-\frac{1}{5}-\frac{1}{21}\right) = \frac{88}{105}$$

25.

obsah časti grafu f-cie $z=f(x,y)$ nad oblasťou $M\subseteq\mathbb{R}^2$

$$S = \iint_M \sqrt{1+(f'_x)^2+(f'_y)^2}\,dx\,dy$$

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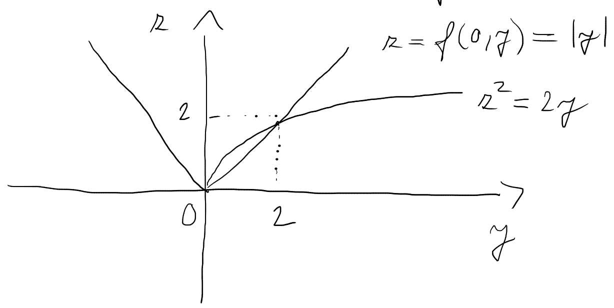
f-cia $z=f(x,y)=\sqrt{x^2+y^2}$; obsah časti jej grafu, ktorá je vyhrýzená paraboloidom $z^2=2y$

$\rightsquigarrow$ priemet do roviny yz :

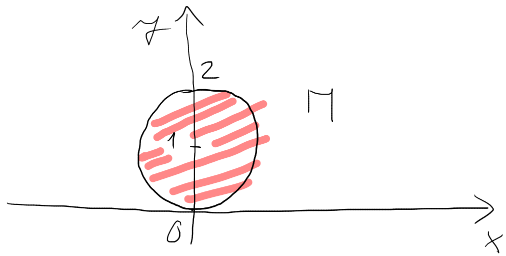


$\rightsquigarrow$ priemet do roviny xy – oblasť M

$z^2 = x^2+y^2 \quad , \quad z^2 = 2y$

$$\Downarrow$$

$$x^2+y^2 = 2y \rightsquigarrow x^2+(y-1)^2 = 1$$



$$f_x' = \frac{x}{\sqrt{x^2+y^2}} \quad , \quad f_y' = \frac{y}{\sqrt{x^2+y^2}} \rightsquigarrow \sqrt{1+(f_x')^2+(f_y')^2} = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} =$$

$$= \sqrt{2}$$

$$S = \iint_M \sqrt{2}\, dx\, dy = \sqrt{2} \cdot \underbrace{\iint_M dx\,dy}$$

obsah oblasti M $\rightsquigarrow$ kruh s polomerom 1

$$\Rightarrow \quad S = \sqrt{2} \cdot \pi \cdot 1^2 = \pi\sqrt{2}$$

APLIKÁCIE TROJNÝCH INTEGRÁLOV

26. objem oblasti $M \subseteq \mathbb{R}^3 \rightsquigarrow V(M) = \iiint_M dx\,dy\,dz$

objem elipsoidu M : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1 \quad , \quad a, b, c > 0$

⇝ transformácia do „sférických súradníc"

$x = a \cdot \varrho \cos\varphi \sin\theta \quad , \quad y = b\varrho \sin\varphi \sin\theta \quad , \quad z = c\varrho\cos\theta$

⇝ jacobián : $J(\varrho, \varphi, \theta) = -abc\varrho^2 \sin\theta$

$M: \quad 0 \le \varphi \le 2\pi \quad ; \quad 0 \le \theta \le \pi \quad ; \quad 0 \le \varrho \le 1$

$$V(M) = \int_0^{2\pi} \left[\int_0^{\pi} \left[\int_0^1 abc\varrho^2 \sin\theta \, d\varrho \right] d\theta \right] d\varphi =$$

$$= abc \cdot \left(\int_0^{2\pi} d\varphi \right) \cdot \left(\int_0^{\pi} \sin\theta \, d\theta \right) \cdot \left(\int_0^1 \varrho^2 d\varrho \right) =$$

$$= abc \left[\varphi \right]_0^{2\pi} \left[-\cos\theta \right]_0^{\pi} \cdot \left[\frac{\varrho^3}{3} \right]_0^1 = abc \cdot 2\pi \cdot 2 \cdot \frac{1}{3}$$

$$= \frac{4}{3}\pi abc$$

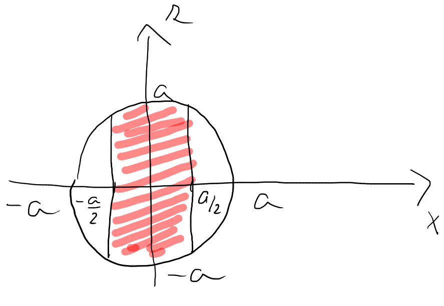
27.

objem oblasti $M \subseteq \mathbb{R}^3$: $x^2+y^2+z^2 = a^2$, $x^2+y^2 = ax$

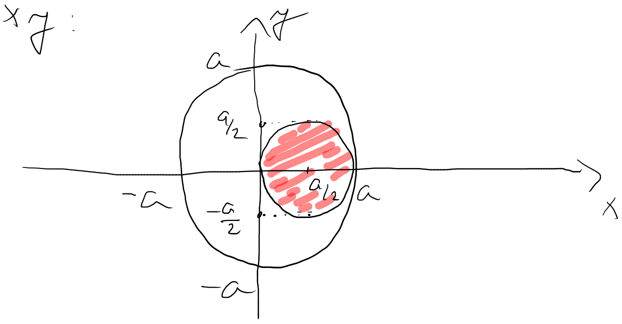
$a > 0$

⇝ $x^2+y^2 = ax \iff \left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$

⇝ priemet do roviny xz :



→ průmět do roviny xy:



⇝ válcové souřadnice: $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$, $J = \varrho$

$M: \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$; $0 \leq \varrho \leq a\cos\varphi$; $-\sqrt{a^2-\varrho^2} \leq z \leq \sqrt{a^2-\varrho^2}$

$$V(M) = \iiint_M dx\,dy\,dz = \int_{-\frac{\pi}{2}}^{\pi/2}\left[\int_0^{a\cos\varphi}\left[\int_{-\sqrt{a^2-\varrho^2}}^{\sqrt{a^2-\varrho^2}} \varrho\, dz\right]d\varrho\right]d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{a\cos\varphi}\left[\varrho z\right]_{-\sqrt{a^2-\varrho^2}}^{\sqrt{a^2-\varrho^2}} d\varrho\right]d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{a\cos\varphi} 2\sqrt{a^2-\varrho^2}\,\varrho\, d\varrho\right]d\varphi =$$

$$= \left| \begin{matrix} t = a^2-\varrho^2 \;, & dt = -2\varrho\, d\varrho \\ 0 \to a^2 \;, & a\cos\varphi \to a^2\sin^2\varphi \end{matrix} \right| = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_{a^2}^{a^2\sin^2\varphi} -\sqrt{t}\, dt\right]d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_{a^2\sin^2\varphi}^{a^2} \sqrt{t}\, dt\right]d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{2}{3}t^{3/2}\right]_{a^2\sin^2\varphi}^{a^2} d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{3}\left(a^3 - a^3|\sin\varphi|^3\right)d\varphi$$

$$= \frac{2}{3}a^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-|\sin\varphi|^3\right)d\varphi = \frac{4}{3}a^3\int_0^{\pi/2}\left(1-\sin^3\varphi\right)d\varphi = \frac{4}{3}a^3\left[\varphi\right]_0^{\frac{\pi}{2}} -$$

$$\frac{4}{3}a^3\int_0^{\pi/2}\sin^3\varphi\, d\varphi = \frac{2}{3}\pi a^3 - \frac{4}{3}a^3\int_0^{\pi/2}\sin^3\varphi = \left| \begin{array}{l} s=\cos\varphi \quad , \; ds=-\sin\varphi\, d\varphi \\ 0 \rightsquigarrow 1 \quad ; \; \frac{\pi}{2} \rightsquigarrow 0 \end{array} \right|$$

$$= \frac{2}{3}\pi a^3 - \frac{4}{3}a^3\int_1^0 (1-s^2)(-ds) = \frac{2}{3}\pi a^3 - \frac{4}{3}a^3\int_0^1 (1-s^2)\, ds =$$

$$= \frac{2}{3}\pi a^3 - \frac{4}{3}a^3\left[s-\frac{1}{3}s^3\right]_0^1 = \frac{2}{3}\pi a^3 - \frac{4}{3}a^3\cdot\frac{2}{3} =$$

$$= \frac{2}{3}a^3\left(\pi-\frac{4}{3}\right) = \frac{2(3\pi-4)}{9}a^3$$