NEVLASTNÉ VIACNÁSOBNÉ INTEGRÁLY

NEOHRANIČENÁ FUNKCIA, FUNKCIA SO SING. BODMI

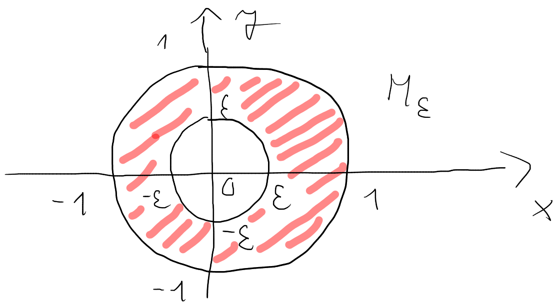
28.

$$I = \iint_M \ln \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy \quad , \quad M: x^2+y^2 \le 1$$

$\rightsquigarrow$ $f(x,y) = \ln \dfrac{1}{\sqrt{x^2+y^2}}$ má v $[0,0]$ singulárny bod;

$$\lim_{(x,y)\to(0,0)} f(x,y) = \infty$$

$\rightsquigarrow$ pre $\varepsilon > 0$:



$M_\varepsilon: \quad \varepsilon^2 \le x^2+y^2 \le 1 \quad ; \quad I_\varepsilon := \iint_{M_\varepsilon} \ln \dfrac{1}{\sqrt{x^2+y^2}}\,dx\,dy$

$\rightsquigarrow$ platí : $I = \lim\limits_{\varepsilon\to 0^+} I_\varepsilon$

$\rightsquigarrow$ polárne súradnice : $x = \rho\cos\varphi \quad , \quad y = \rho\sin\varphi \quad , \quad J = \rho$

$M_\varepsilon: \quad 0 \le \varphi \le 2\pi \quad ; \quad \varepsilon \le \rho \le 1$

$$I_\varepsilon = \int_0^{2\pi} \left[\int_\varepsilon^1 \left(\ln \frac{1}{\rho}\right) \cdot \rho \, d\rho \right] d\varphi = \left(\int_0^{2\pi} d\varphi \right) \cdot \left(\int_\varepsilon^1 \rho \ln \frac{1}{\rho} \, d\rho \right) =$$

$$= 2\pi \int_\varepsilon^1 -\rho \ln\rho \, d\rho = -2\pi \int_\varepsilon^1 \rho \ln\rho \, d\rho = \begin{vmatrix} u' = \rho & , \ v = \ln\rho \\ u = \frac{\rho^2}{2} & , \ v' = \frac{1}{\rho} \end{vmatrix}$$

$$= -2\pi \left(\left[\frac{\rho^2}{2} \ln\rho \right]_\varepsilon^1 - \int_\varepsilon^1 \frac{\rho}{2} \, d\rho \right) = -2\pi \left(-\frac{\varepsilon^2}{2} \ln\varepsilon - \left[\frac{\rho^2}{4} \right]_\varepsilon^1 \right)$$

$$= 2\pi \left(\frac{\varepsilon^2}{2} \ln\varepsilon + \frac{1 - \varepsilon^2}{4} \right)$$

$$I = \lim_{\varepsilon \to 0^+} I_\varepsilon = \lim_{\varepsilon \to 0^+} 2\pi \left(\frac{\varepsilon^2}{2} \ln\varepsilon + \frac{1 - \varepsilon^2}{4} \right) =$$

$$= \pi \lim_{\varepsilon \to 0^+} \left(\varepsilon^2 \ln\varepsilon \right) + \frac{\pi}{2}$$

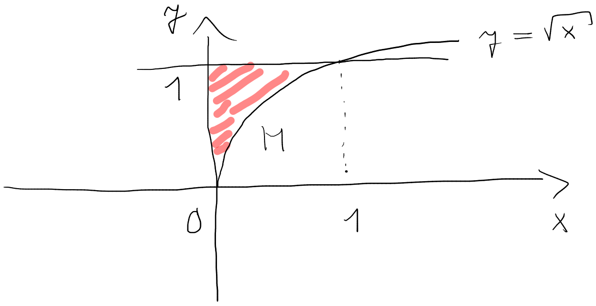
$\leadsto$ platí $\displaystyle \lim_{\varepsilon \to 0^+} \varepsilon^2 \ln\varepsilon = (0 \cdot \infty) = \lim_{\varepsilon \to 0^+} \frac{\ln\varepsilon}{\frac{1}{\varepsilon^2}} = \left(\frac{\infty}{\infty} \right)$

$$\overset{\text{L'H}}{=} \lim_{\varepsilon \to 0^+} \frac{\frac{1}{\varepsilon}}{-\frac{2}{\varepsilon^3}} = \lim_{\varepsilon \to 0^+} -\frac{\varepsilon^2}{2} = 0 \ , \text{ proto}$$

$$\boxed{I = \frac{\pi}{2}}$$

29.

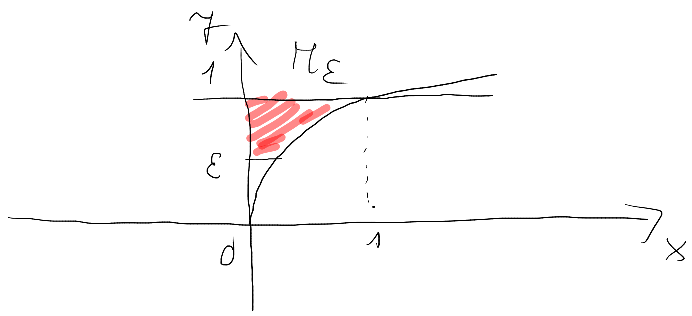
$$I = \iint_M e^{\frac{x}{y}}\,dx\,dy \quad ; \quad M: \ y \le 1 \ , x \ge 0 \ , y \ge \sqrt{x}$$



$\rightsquigarrow$ $f(x,y) = e^{\frac{x}{y}}$ má v $[0,0]$ singulárny bod

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} e^{\frac{x}{y}} = \text{neexistuje}$$

$\rightsquigarrow$ pre $\varepsilon > 0$



M_ε: $\quad \varepsilon \le y \le 1 \quad ; \quad 0 \le x \le y^2 \quad ; \quad I_\varepsilon := \iint_{M_\varepsilon} e^{\frac{x}{y}}\,dx\,dy$

$\rightsquigarrow$ platí: $I = \lim_{\varepsilon \to 0^+} I_\varepsilon$

$$I_\varepsilon = \int_\varepsilon^1 \left[\int_0^{y^2} e^{\frac{x}{y}}\,dx \right] dy = \int_\varepsilon^1 \left[y e^{\frac{x}{y}} \right]_0^{y^2} dy$$

$$= \int_\varepsilon^1 (y e^y - y)\,dy = \int_\varepsilon^1 y e^y\,dy - \left[\tfrac{y^2}{2} \right]_\varepsilon^1 =$$

$$= \int_{\varepsilon}^{1} y e^{y}\, dy - \frac{1-\varepsilon^2}{2} = \begin{vmatrix} u = y & , & v' = e^{y} \\ u' = 1 & , & v = e^{y} \end{vmatrix}$$

$$= \left[y e^{y} \right]_{\varepsilon}^{1} - \int_{\varepsilon}^{1} e^{y}\, dy - \frac{1-\varepsilon^2}{2} = e - \varepsilon e^{\varepsilon} - \frac{1-\varepsilon^2}{2}$$

$$- \left[e^{y} \right]_{\varepsilon}^{1} = e - \varepsilon e^{\varepsilon} - \frac{1-\varepsilon^2}{2} - (e - e^{\varepsilon}) = e^{\varepsilon}(1-\varepsilon) - \frac{1-\varepsilon^3}{2}$$

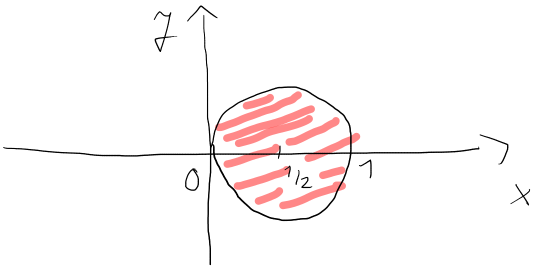
$$\rightsquigarrow I = \lim_{\varepsilon \to 0^+} I_{\varepsilon} = \lim_{\varepsilon \to 0^+} \left(e^{\varepsilon}(1-\varepsilon) - \frac{1-\varepsilon^2}{2} \right) = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

30.

$$I = \iint_{M} \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy \quad , \quad M: \ x^2 + y^2 \leq x$$

$$\Updownarrow$$

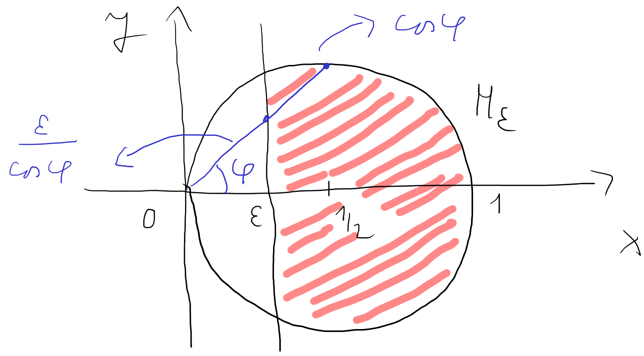
$$\left(x - \frac{1}{2}\right)^2 + y^2 \leq \left(\frac{1}{2}\right)^2$$



$\rightsquigarrow$ $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ má v $[0,0]$ singulárny bod

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{x^2+y^2}} = \infty$$

$\rightsquigarrow$ pre $\varepsilon \in (0,1)$



polárne súradnice

$x = \rho\cos\varphi \ , \ y = \rho\sin\varphi$

$J = \rho$

$$M_\varepsilon : \ \frac{\varepsilon}{\cos\varphi} \le \rho \le \cos\varphi \ ; \quad -\arccos\sqrt{\varepsilon} \le \varphi \le \arccos\sqrt{\varepsilon}$$

$$\left(\frac{\varepsilon}{\cos\varphi} = \cos\varphi \rightsquigarrow \cos\varphi = \sqrt{\varepsilon} \right)$$

$$\rightsquigarrow \ I = \lim_{\varepsilon\to 0^+} I_\varepsilon$$

$$I_\varepsilon = \int_{-\arccos\sqrt{\varepsilon}}^{\arccos\sqrt{\varepsilon}} \left[\int_{\frac{\varepsilon}{\cos\varphi}}^{\cos\varphi} \frac{1}{\rho}\cdot\rho \, d\rho \right] d\varphi = \int_{-\arccos\sqrt{\varepsilon}}^{\arccos\sqrt{\varepsilon}} \Big[\rho \Big]_{\varepsilon/\cos\varphi}^{\cos\varphi} d\varphi =$$

$$= \int_{-\arccos\sqrt{\varepsilon}}^{\arccos\sqrt{\varepsilon}} \left(\cos\varphi - \frac{\varepsilon}{\cos\varphi} \right) d\varphi = 2\cdot\int_0^{\arccos\sqrt{\varepsilon}} \frac{\cos^2\varphi - \varepsilon}{\cos\varphi} \, d\varphi =$$

$$= 2\int_0^{\arccos\sqrt{\varepsilon}} \frac{1-\sin^2\varphi-\varepsilon}{\cos^2\varphi} \cos\varphi \, d\varphi = 2\int_0^{\arccos\sqrt{\varepsilon}} \frac{1-\sin^2\varphi-\varepsilon}{1-\sin^2\varphi}\cdot\cos\varphi \, d\varphi$$

$$= \left| \begin{array}{l} t = \sin\varphi \ ; \ dt = \cos\varphi \, d\varphi \\ 0 \rightsquigarrow 0 \ , \ \arccos\sqrt{\varepsilon} \rightsquigarrow \sin(\arccos\sqrt{\varepsilon}) = \sqrt{1-\varepsilon} \end{array} \right| =$$

$$\int_0^{\sqrt{1-\varepsilon}} \frac{1-t^2-\varepsilon}{1-t^2}\,dt = \int_0^{\sqrt{1-\varepsilon}} \left(1 + \frac{\varepsilon}{t^2-1}\right)dt = \int_0^{\sqrt{1-\varepsilon}} \left(1 + \frac{\varepsilon/2}{t-1} - \frac{\varepsilon/2}{t+1}\right)dt$$

$$= \left[t + \frac{\varepsilon}{2} \ln\left|\frac{t-1}{t+1}\right| \right]_0^{\sqrt{1-\varepsilon}} = \sqrt{1-\varepsilon} + \frac{\varepsilon}{2} \ln\left|\frac{\sqrt{1-\varepsilon}-1}{\sqrt{1-\varepsilon}+1}\right| =$$

$$= \sqrt{1-\varepsilon} + \frac{\varepsilon}{2} \ln\left(\frac{1-\sqrt{1-\varepsilon}}{1+\sqrt{1-\varepsilon}}\right)$$

$$\Rightarrow \quad I = \lim_{\varepsilon \to 0^+} \left(\sqrt{1-\varepsilon} + \frac{\varepsilon}{2} \ln\left(\frac{1-\sqrt{1-\varepsilon}}{1+\sqrt{1-\varepsilon}}\right)\right) \overset{\sigma := \sqrt{1-\varepsilon}}{=}$$

$$= \lim_{\sigma \to 1^-} \left(\sigma + \frac{1-\sigma^2}{2} \ln\left(\frac{1-\sigma}{1+\sigma}\right)\right) =$$

$$= 1 + \frac{1}{2} \lim_{\sigma \to 1^-} (1-\sigma^2) \ln\left(\frac{1-\sigma}{1+\sigma}\right) ;$$

$$\lim_{\sigma \to 1^-} (1-\sigma^2) \ln\left(\frac{1-\sigma}{1+\sigma}\right) = (0 \cdot \infty) = \lim_{\sigma \to 1^-} \frac{\ln\left(\frac{1-\sigma}{1+\sigma}\right)}{\frac{1}{1-\sigma^2}} = \left(\frac{\infty}{\infty}\right)$$

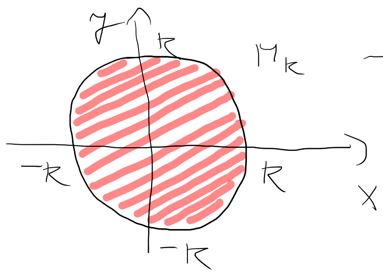
$$\overset{\text{L'H}}{=} \lim_{\sigma \to 1^-} \frac{\frac{-2}{1-\sigma^2}}{\frac{2\sigma}{(1-\sigma^2)^2}} = \lim_{\sigma \to 1^-} \frac{\sigma^2-1}{\sigma} = 0$$

$$\Rightarrow \boxed{I = 1}$$

NEOHRANIČENÝ INTEGRAČNÝ OBOR

31. $$I = \iint_M \frac{1}{1+x^2+y^2}\,dx\,dy \quad , \quad M = \mathbb{R}^2$$

pre $R > 0$:



M_R → polárne súradnice

$$x = \rho\cos\varphi \quad , \quad y = \rho\sin\varphi$$

$$M_R: \quad 0 \le \varphi \le 2\pi$$
$$\quad\quad 0 \le \rho \le R$$

$$I_R := \iint_{M_R} \frac{1}{1+x^2+y^2}\,dx\,dy \quad ; \quad I = \lim_{R\to\infty} I_R$$

$$\rightsquigarrow \quad I_R = \int_0^{2\pi}\left[\int_0^R \frac{1}{1+\rho^2}\,\rho\,d\rho\right]d\varphi = \int_0^{2\pi}\left[\frac{1}{2}\ln(1+\rho^2)\right]_0^R d\varphi$$

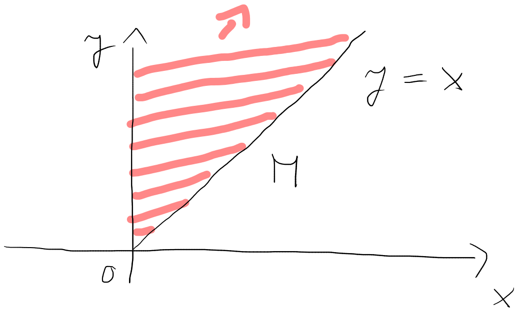
$$= \int_0^{2\pi} \frac{1}{2}\ln(1+R^2)\,d\varphi = \frac{1}{2}\ln(1+R^2)\left[\varphi\right]_0^{2\pi} = \pi\ln(1+R^2)$$

$$\rightsquigarrow \quad I = \lim_{R\to\infty} I_R = \lim_{R\to\infty} \pi\ln(1+R^2) = \infty$$

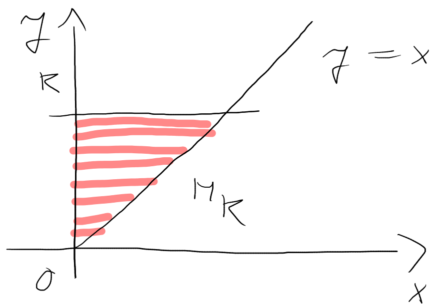
integrál I diverguje

32.

$$I = \iint\limits_{M} e^{-(x+y)}\,dx\,dy \quad ; \quad M:\ 0 \le x \le y\ ,\ y \ge 0$$



pre $k > 0$ $\rightsquigarrow$



$$M_k:\ 0 \le x \le y \quad ; \quad 0 \le y \le k$$

$$I_k := \iint\limits_{M_k} e^{-(x+y)}\,dx\,dy \quad , \ \text{platí}:\ I = \lim_{k\to\infty} I_k$$

$$\rightsquigarrow\ I_k = \int_0^k \left[\int_0^y e^{-(x+y)}\,dx \right] dy = \int_0^k \left[-e^{-(x+y)} \right]_0^y dy$$

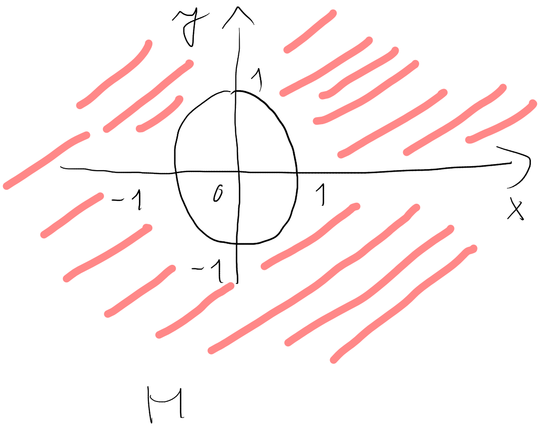
$$= \int_0^k \left(e^{-y} - e^{-2y} \right) dy = \left[-e^{-y} + \frac{1}{2} e^{-2y} \right]_0^k$$

$$= -e^{-k} + \frac{1}{2} e^{-2k} + \frac{1}{2}$$

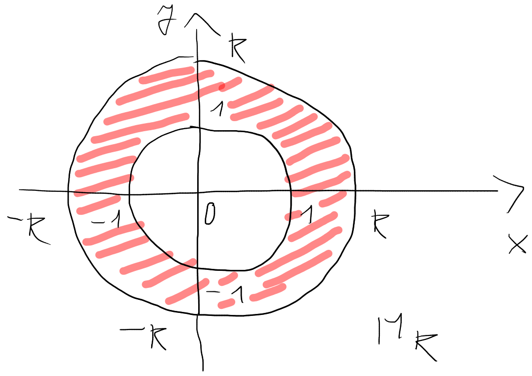
$$I = \lim_{k\to\infty} I_k = \lim_{k\to\infty} \left(-e^{-k} + \frac{1}{2} e^{-2k} + \frac{1}{2} \right) = \boxed{\frac{1}{2}}$$

33.

$$I = \iint\limits_{M} \frac{1}{(x^2+y^2)^\alpha}\, dx\, dy \quad ; M: \; x^2+y^2 \geq 1 \quad ; \alpha > 1$$



$\xrightarrow{R>1}$



$\rightsquigarrow$ platí : $I = \lim\limits_{R\to\infty} I_R$, kde $I_R = \iint\limits_{M_R} \frac{1}{(x^2+y^2)^\alpha}\, dx\, dy$

polární souřadnice : $x = \rho\cos\varphi$, $y = \rho \sin\varphi$; $J = \rho$

M_R: $\quad 0 \leq \varphi \leq 2\pi \quad ; \quad 1 \leq \rho \leq R$

$$I_R = \int_0^{2\pi} \left[\int_1^R \frac{\rho}{\rho^{2\alpha}}\, d\rho \right] d\varphi = \left(\int_0^{2\pi} d\varphi \right) \cdot \left(\int_1^R \rho^{1-2\alpha}\, d\rho \right)$$

$$= [\varphi]_0^{2\pi} \cdot \left[\frac{1}{2-2\alpha} \cdot \rho^{2-2\alpha} \right]_1^R = 2\pi \cdot \frac{1}{2-2\alpha} \cdot \left(R^{2-2\alpha} - 1 \right)$$

$$= \frac{\pi}{1-\alpha} \cdot \left(R^{2(1-\alpha)} - 1 \right)$$

$$I = \lim_{R\to\infty} I_R = \lim_{R\to\infty} \frac{\pi}{1-\alpha}\left(R^{2(1-\alpha)} - 1\right) = \boxed{\frac{\pi}{\alpha-1}}$$

(34.) aplikácia ⇝ výpočet POISSONOVHO INTEGRÁLU

$$\underline{I = \int_{-\infty}^{\infty} e^{-x^2}\,dx}$$

⇝ konvergencia I?

platí: $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \underbrace{\frac{x^4}{2} + \frac{x^6}{6} + \dots}_{\geq 0}$ pre $\forall x \in \mathbb{R}$

$\Rightarrow \quad e^{x^2} \geq 1 + x^2$ pre $\forall x \in \mathbb{R}$

$\Leftrightarrow 0 < e^{-x^2} \leq \frac{1}{1+x^2}$ pre $\forall x \in \mathbb{R}$

⇝ $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx \leq \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \left[\operatorname{arctg} x\right]_{-\infty}^{\infty} = \pi$

$\Rightarrow$ <u>I konverguje</u>; $I \leq \pi$

⇝ platí: $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2-y^2} \, dy \right] dx = \underbrace{\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy}_{\text{integrál existuje}}$$

$\rightsquigarrow$ polárne súradnice :

$x = \rho \cos\varphi$, $y = \rho \sin\varphi$; $J = \rho$;

$0 \le \varphi \le 2\pi$; $0 \le \rho < \infty$

$$\rightsquigarrow \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dx \, dy = \int_0^{2\pi} \left[\int_0^{\infty} e^{-\rho^2} \rho \, d\rho \right] d\varphi =$$

$$= \left| \begin{array}{ll} t = -\rho^2 ; & dt = -2\rho \, d\rho \\ 0 \rightsquigarrow 0 ; & \infty \rightsquigarrow -\infty \end{array} \right| = \int_0^{2\pi} \left[\int_0^{-\infty} -\frac{e^t}{2} \, dt \right] d\varphi$$

$$= \int_0^{2\pi} \left[\int_{-\infty}^{0} \frac{e^t}{2} \, dt \right] d\varphi = \int_0^{2\pi} \left[\frac{e^t}{2} \right]_{-\infty}^{0} d\varphi = \int_0^{2\pi} \frac{1}{2} \left(1 - \lim_{t \to -\infty} e^t \right) d\varphi$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\varphi = \left[\frac{\varphi}{2} \right]_0^{2\pi} = \pi$$, teda:

$$I^2 = \pi \quad \Rightarrow \quad I = \sqrt{\pi}$$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}}$$