D.V.1 1) $n=1$; $U_1 \sim N(0,1)$, $f_{U_1^2} = ?$

$u \geq 0$: $F_{U_1^2}(u) = P(U_1^2 \leq u) = P(|U_1| \leq \sqrt{u}) = P(-\sqrt{u} \leq U_1 \leq \sqrt{u})$

$$= F_{U_1}(\sqrt{u}) - F_{U_1}(-\sqrt{u})$$

$$f_{U_1^2}(u) = f_{U_1}(\sqrt{u}) \cdot \frac{1}{2} u^{-\frac{1}{2}} + f_{U_1}(-\sqrt{u}) \cdot \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2} u^{-\frac{1}{2}} \left(f_{U_1}(\sqrt{u}) + f_{U_1}(-\sqrt{u}) \right)$$

$$\boxed{f_{U_1}(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}} \qquad \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

$$f_{U_1^2}(u) = \frac{1}{2} u^{-\frac{1}{2}} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u} \right] = \frac{1}{\sqrt{2\pi}} u^{-\frac{1}{2}} e^{-\frac{1}{2}u}$$

$$= \frac{1}{2^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)} u^{\frac{1}{2}-1} e^{-\frac{1}{2}u} \sim \chi^2(1)$$

2) Předp., že V. platí pro $n \geq 1$ a dokážeme pro $n+1$. $U_1, \dots, U_{n+1} \sim N(0,1)$, nezávislé

$$f_{\underbrace{U_1^2 + \dots + U_n^2}_{\chi^2(n)} + \underbrace{U_{n+1}^2}_{\chi^2(1)}}(u) = \int_{-\infty}^{\infty} f_{U_1^2 + \dots + U_n^2}(u-x) \cdot f_{U_{n+1}^2}(x)\, dx$$

$$= \int_0^{u} \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} (u-x)^{\frac{n}{2}-1} e^{-\frac{1}{2}(u-x)} \cdot \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{n}{2})} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x}\, dx$$

$x \geq 0$

$u - x \geq 0$

$u \geq x$

$$= \int_0^{u} \frac{1}{2^{\frac{n+1}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})} e^{-\frac{1}{2}u} (u-x)^{\frac{n}{2}-1} x^{-\frac{1}{2}}\, dx$$

$$= \frac{1}{2^{\frac{n+1}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})} e^{-\frac{1}{2}u} \int_0^{u} u^{\frac{n}{2}-1-\frac{1}{2}} \left(1-\frac{x}{u}\right)^{\frac{n}{2}-1} \left(\frac{x}{u}\right)^{-\frac{1}{2}} dx$$

$$\boxed{\begin{aligned} t &= \frac{x}{u} \\ dt &= \frac{1}{u}dx \end{aligned}}$$

$$= \frac{e^{-\frac{1}{2}u}\, u^{\frac{n-3}{2}} \cdot u}{2^{\frac{n+1}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})} \underbrace{\int_0^{1} (1-t)^{\frac{n}{2}-1} t^{-\frac{1}{2}}\, dt}_{B(\frac{1}{2};\frac{n}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}}$$

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt$$

$$B(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{1}{2^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})} u^{\frac{n+1}{2}-1} e^{-\frac{1}{2}u} \sim \chi^2(n+1)$$

D.V.S.

$$T = \frac{U}{\sqrt{\frac{k}{\nu}}} = \frac{\sqrt{\nu}\,U}{\sqrt{k}} \overset{V.12.10.}{\Longrightarrow} f_T(u) = \frac{1}{\sqrt{\nu}} \int_0^\infty f_U\left(\frac{u x}{\sqrt{\nu}}\right) \cdot f_{\sqrt{k}}(x) \cdot x\,dx$$

$$f_k(x) = \begin{cases} \frac{1}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{1}{2}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$y \geq 0 \qquad F_{\sqrt{k}}(y) = P(\sqrt{k} \leq y) = \underbrace{P(k \leq y^2)}_{F_k(y^2)} = \int_0^{y^2} \frac{1}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{1}{2}x} dx$$

$$x = t^2$$
$$dx = 2t\,dt$$

$$= \int_0^y \underbrace{\frac{1}{2^{\frac{\nu}{2}-1} \Gamma\left(\frac{\nu}{2}\right)} t^{\nu-2+1} e^{-\frac{1}{2}t^2}}_{f_{\sqrt{k}}(t)} dt$$

$$f_T(u) = \frac{1}{\sqrt{\nu}} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{u^2 x^2}{\nu}} \cdot \frac{1}{2^{\frac{\nu}{2}-1} \Gamma\left(\frac{\nu}{2}\right)} x^{\nu} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{1}{\sqrt{\pi\nu}\, \Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu-1}{2}}} \int_0^\infty e^{-\frac{x^2}{2}\left(\frac{u^2}{\nu}+1\right)} x^{\nu}\,dx$$

$$= \frac{1}{\sqrt{\pi\nu}\,\Gamma(\frac{\nu}{2})\,2^{\frac{\nu-1}{2}}} \int_0^\infty e^{-\frac{x^2}{2}\left(\frac{u^2}{\nu}+1\right)} \underset{x^{\nu-1}\,x\,dx}{x^\nu\,dx}$$

$$t = \frac{x^2}{2}\left(\frac{u^2}{\nu}+1\right)$$

$$dt = x\,dx\left(\frac{u^2}{\nu}+1\right)$$

$$x = \frac{(2t)^{\frac{1}{2}}}{\left(\frac{u^2}{\nu}+1\right)^{\frac{1}{2}}}$$

$$\frac{2^{\frac{\nu-1}{2}}}{\Gamma(\frac{\nu}{2})\Gamma(\frac{1}{2})\sqrt{\nu}\,2^{\frac{\nu-1}{2}}\left(\frac{u^2}{\nu}+1\right)^{\frac{\nu-1}{2}+1}} \underbrace{\int_0^\infty e^{-t}\,t^{\frac{\nu+1}{2}-1}\,dt}_{\Gamma\left(\frac{\nu+1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\frac{\nu}{2})\Gamma(\frac{1}{2})}\,\nu^{-\frac{1}{2}}\left(\frac{u^2}{\nu}+1\right)^{-\frac{\nu+1}{2}} \sim t(\nu+1)$$

D.V.7.

$$F = \frac{\frac{k_1}{\nu_1}}{\frac{k_2}{\nu_2}} = \frac{\frac{\nu_2}{\nu_1}\cdot k_1}{k_2} \quad ; \quad f_{k_i}(x_i) = \begin{cases} \frac{1}{2^{\frac{\nu_i}{2}}\,\Gamma(\frac{\nu_i}{2})}\,x_i^{\frac{\nu_i}{2}-1}\,e^{-\frac{1}{2}x_i} & x_i \geq 0 \\ 0 & x_i < 0 \end{cases}$$

$$f_F(u) = \frac{\nu_1}{\nu_2}\int_0^\infty f_{k_1}\left(\frac{u\,x\,\nu_1}{\nu_2}\right) f_{k_2}(x)\,x\,dx$$

$$f_F(u) = \frac{\nu_1}{\nu_2} \int_0^\infty \frac{1}{2^{\frac{\nu_1}{2}} \Gamma\left(\frac{\nu_1}{2}\right)} \left(\frac{u x \nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}-1} e^{-\frac{1}{2}\left(\frac{u x \nu_1}{\nu_2}\right)} \cdot \frac{1}{2^{\frac{\nu_2}{2}} \Gamma\left(\frac{\nu_2}{2}\right)} x^{\frac{\nu_2}{2}-1} e^{-\frac{1}{2}x} \, x \, dx$$

$$= \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{u^{\frac{\nu_1}{2}-1}}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right)} \int_0^\infty x^{\frac{\nu_1+\nu_2}{2}-1} e^{-\frac{1}{2}x\left(\frac{u\nu_1}{\nu_2}+1\right)} dx$$

$$t = \frac{x}{2}\left(\frac{u\nu_1}{\nu_2}+1\right) \Rightarrow x = \frac{2t}{\left(\frac{u\nu_1}{\nu_2}+1\right)}$$

$$dt = \frac{1}{2}\left(\frac{u\nu_1}{\nu_2}+1\right) dx$$

$$= \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{2^{\frac{\nu_1+\nu_2}{2}} \, u^{\frac{\nu_1}{2}-1}}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right) \left(\frac{u\nu_1}{\nu_2}+1\right)^{\frac{\nu_1+\nu_2}{2}-1+1}} \underbrace{\int_0^\infty t^{\frac{\nu_1+\nu_2}{2}-1} e^{-t} dt}_{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} u^{\frac{\nu_1}{2}-1} \left(\frac{\nu_1}{\nu_2}u+1\right)^{-\frac{\nu_1+\nu_2}{2}} \sim F(\nu_1, \nu_2)$$