Borelovské množiny $(\Omega, \mathcal{A})$

<u>V.</u> S systém podmnožin Ω. Pak existuje σ-algebra $\sigma(S)$ tak, že

1) $S \subseteq \sigma(S)$

2) $\mathcal{A}^*$ je σ-algebra taková, že $S \subseteq \mathcal{A}^* \Rightarrow \sigma(S) \subseteq \mathcal{A}^*$

$\sigma(S)$ ... <u>minimální σ-algebra generovaná</u> S

$$\Omega = (-\infty, \infty) = \mathbb{R}$$

$$S = \{(-\infty, x\rangle \, ; \, x \in \mathbb{R}\} \, , \quad S \subseteq 2^{\Omega}$$

$$\exists \; \sigma(S) = \mathcal{B} \; \text{....} \; \underline{\text{borelovská } \sigma\text{-algebra}}$$

$$A \in \mathcal{B} \Rightarrow A \text{ ... borelovská množina}$$

$$\Omega = \mathbb{R}^n \, , \quad S = \{(-\infty, x_1\rangle \times \dots \times (-\infty, x_n) \, ; \, (x_1, \dots, x_n) \in \mathbb{R}^n\}$$

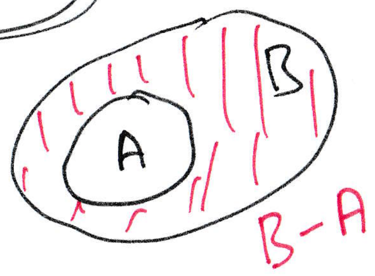
$$\sigma(S) = \mathcal{B}^n$$

D.V.2

1) $\underline{P(\Omega)=1} = P(\Omega \cup \emptyset \cup \emptyset \dots) = \underbrace{P(\Omega)}_{1} + \underbrace{P(\emptyset)}_{0} + \underbrace{\dots}_{0} = 1$

2) $P(A \cup B) = P(A \cup B \cup \emptyset \cup \emptyset \cup \dots) = \underline{P(A)+P(B)} + \underbrace{P(\emptyset)}_{0} + \underbrace{\dots}_{0 \dots}$

3) $\boxed{A \subseteq B}$



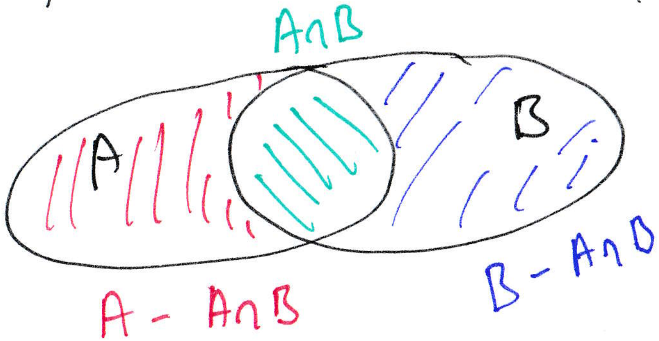
$B = A \cup (B-A)$

$\underline{P(B)} = \underline{P(A)} + \underbrace{P(B-A)}_{\geq 0}$

4) $P(B) \geq P(A)$

5), 6) $\bar{A} = \Omega - A$, $A \subseteq \Omega$, $P(\bar{A}) = P(\Omega - A) = \underbrace{P(\Omega)}_{1} - P(A)$

7)



$P(A \cup B) = P(\underbrace{A - A\cap B}_{A\cap B \subseteq A}) + P(A \cap B) + P(\underbrace{B - A\cap B}_{B \supseteq A\cap B})$

$= P(A) - P(A\cap B) + \cancel{P(A\cap B)} + P(B) - \cancel{P(A\cap B)}$

$n \geq 3$

$$P\Big(\underbrace{\bigcup_{i=1}^{n-1} A_i}_{B}\Big) = \underline{\sum_{i=1}^{n-1} P(A_i)} - \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} P(A_i \cap A_j) - \dots (-1)^{n-2} P(A_1 \cap \dots \cap A_{n-1})$$

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = P\Big(\underbrace{\bigcup_{i=1}^{n-1} A_i}_{B} \cup A_n\Big) = \underline{P(B)} + \underline{P(A_n)} - P(B \cap A_n)$$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} P(A_i \cap A_j) - \dots (-1)^{n-2} P(A_1 \cap \dots \cap A_{n-1}) -$$

$$- P\Big(\bigcup_{i=1}^{n-1} (A_i \cap A_n)\Big)$$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} P(A_i \cap A_j) - \dots (-1)^{n-2} P(A_1 \cap \dots \cap A_{n-1})$$

$$- \sum_{i=1}^{n-1} P(A_i \cap A_n) + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} P(A_i \cap A_n \cap A_j) - \dots (-1)^{n-1} \underline{P(A_1 \cap \dots A_{n-1} \cap A_n)}$$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P(A_i \cap A_j) - \dots - (-1)^{n-1} P(A_1 \cap \dots \cap A_n)$$

9) $$P\Big(\bigcup_{i=1}^{n} A_i\Big) = P\Big(\bigcup_{i=1}^{n-1} A_i \cup A_n\Big) = P\Big(\bigcup_{i=1}^{n-1} A_i\Big) + P(A_n) - P\Big(\bigcup_{i=1}^{n-1} A_i \cap A_n\Big)$$

$$= \underline{P\Big(\bigcup_{i=1}^{n-2} A_i\Big)} + P(A_{n-1}) \underbrace{- P\Big(\bigcup_{i=1}^{n-2} A_i \cap A_{n-1}\Big)}_{\geq 0} + P(A_n) \underbrace{- P\Big(\bigcup_{i=1}^{n-1} A_i \cap A_n\Big)}_{\geq 0}$$

$\vdots$

$$\geq \sum_{i=1}^{n} P(A_i)$$

D.V.3. $1) \Rightarrow 2)$ $\ldots \subseteq A_n \subseteq A_{n+1} \subseteq \ldots$ $\qquad \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$

$\ldots$

$$A_1 \cup A_2 \cup A_3 \cup \ldots = \underbrace{A_1}_{B_1} \cup \underbrace{(A_2 - A_1)}_{B_2} \cup \underbrace{(A_3 - A_2)}_{B_3} \cup \ldots$$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \qquad B_n = A_n - A_{n-1}$$

$$\underline{P(\lim_{n\to\infty} A_n)} = P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = P\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} P(B_n) = \cancel{P(A_1)} + \cancel{P(A_2)} - \cancel{P(A_1)} + P(A_3) - \cancel{P(A_2)} + \ldots$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} P(B_n) = \underline{\lim_{k\to\infty} P(A_k)}$$

2) $\Rightarrow$ 3) $\ldots \supseteq A_n \supseteq A_{n+1} \supseteq \ldots \qquad \ldots \subseteq \overline{A_n} \subseteq \overline{A_{n+1}} \subseteq \ldots$

$$\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} 1 - P(\overline{A_n}) = 1 - \lim_{n\to\infty} P(\overline{A_n}) = 1 - P(\lim_{n\to\infty} \overline{A_n})$$

$$= 1 - P\left(\bigcup_{n=1}^{\infty} \overline{A_n}\right) = 1 - P\left(\overline{\bigcap_{n=1}^{\infty} A_n}\right) = 1 - \left(1 - P\left(\bigcap_{n=1}^{\infty} A_n\right)\right)$$

$$= P\left(\bigcap_{n=1}^{\infty} A_n\right) = P(\lim_{n\to\infty} A_n)$$

3) $\Rightarrow$ 4) $\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right) = P(\emptyset) = 0$

4) $\Rightarrow$ 1) Chceme $B_1, \ldots, B_n, \ldots$ $\underline{B_i \cap B_j = \emptyset}$ $P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n)$

$Z_n = \bigcup_{i=n}^{\infty} B_i \quad ; \; \ldots \supseteq Z_n \supseteq Z_{n+1} \supseteq \ldots \qquad \lim_{n\to\infty} Z_n = \bigcap_{n=1}^{\infty} Z_n = \emptyset$

$\bigcap_{n=1}^{\infty} Z_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B_i = \limsup_{n\to\infty} B_n = \{w ; \; w$ patří do nekoneč. mnoha $B_n\}$
$\emptyset$

$$P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_1 \cup \ldots \cup B_n \cup Z_{n+1}) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} P(B_i) + P(Z_{n+1})\right)$$

$$= \sum_{i=1}^{\infty} P(B_i) + \underbrace{\lim_{n\to\infty} P(Z_{n+1})}_{0}$$

D.V.4. $\exists \lim\limits_{n\to\infty} A_n \overset{\text{Chceme}}{\Longrightarrow} P(\lim\limits_{n\to\infty} A_n) = \lim\limits_{n\to\infty} P(A_n)$

$$\bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k=n}^{\infty} A_k}_{B_n} = \liminf_{n\to\infty} A_n = \lim A_n = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} A_k}_{C_n}$$

$$\ldots \subseteq B_n \subseteq B_{n+1} \subseteq \ldots \qquad\qquad \ldots \supseteq C_n \supseteq C_{n+1} \supseteq \ldots$$

$$P(\lim_{n\to\infty} B_n) = P(\lim_{n\to\infty} A_n) = P(\lim_{n\to\infty} C_n)$$

$$\lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} P(C_n)$$

$$B_n \subseteq A_n \subseteq C_n \Rightarrow \underline{P(B_n) \le P(A_n) \le P(C_n)}$$