M6140 Topology Exercises - 1st Week (2020)

1 Closed Sets

Exercise 1. Prove that an arbitrary intersection of closed sets is closed and that a finite union of closed sets is closed.

Exercise 2. Show that a subset F of a topological space X is closed iff for each $x \notin F$ there exists an open set $U \ni x$ such that $U \cap F = \emptyset$.

Exercise 3. Let A, B be arbitrary subsets of a topological space X. Prove the following properties of the closure.

(i) $\overline{\emptyset} = \emptyset$,

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

(iii) $A \subseteq \overline{A}$,

(iv) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$,

(v) $\overline{\overline{A}} = \overline{A}$.

An operator on an arbitrary power set is called a *closure operator* if it satisfies the properties (iii), (iv), (v), thus we now know that $\overline{(-)} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator.

2 Topologies

Exercise 4. An *Alexandrov topology* is a topology in which an arbitrary intersection of open sets is always open.

(a) Let X be a preordered set[1]. Prove that there is an Alexandrov topology on X such that the open sets in X are precisely the lower sets[2] in X.

(b) Let X be a topological space whose topology is Alexandrov. Prove that there is a preorder $\leq$ on X defined by: $x \leq y$ iff $y \in \overline{\{x\}}$.

(c) Prove that these two correspondences are inverse to each other.

Exercise 5. Consider the subsets of $\mathbb{Z}$ of the form $S(a, b) := \{an + b \mid n \in \mathbb{Z}\}$ of $\mathbb{Z}$, where a is a non-zero integer and b is an integer. Define a subset U of $\mathbb{Z}$ to be open iff for each $b \in U$ there exists a non-zero integer a such that $S(a, b) \subseteq U$.

[1] A *preorder* is a reflexive and transitive relation.

[2] A *lower set* in a preordered set is a subset such that if an element belongs to the subset, then all the lower elements also belong to the subset.

(a) Show that this defines a topology on $\mathbb{Z}$. This topology is called the *evenly spaced integer topology* or the *Furstenberg topology*.

(b) Show that each set $S(a, b)$ is clopen.

(c) Show that each open set is either empty or infinite.

(d) Show that the complement of $\{-1, 1\}$ is $\bigcup_{p \text{ prime}} S(p, 0)$.

(e) Conclude that there exist infinitely many primes.

Exercise 6. Suppose that P is a poset. Define a subset U of P to be open iff it is an upper set and each directed set[3] in P whose supremum belongs to U has a non-empty intersection with U.

(a) Show that this defines a topology on P. This topology is called the *Scott topology*.

(b) Show that a subset of P is closed iff it is a lower set that is closed under directed suprema in P.

(c) Show that a mapping $P \to Q$ between posets is continuous iff it preserves directed suprema.

Exercise 7. Let $\mathbb{k}$ be an algebraically closed field[4] and let n be a positive integer. Define a subset F of $\mathbb{k}^n$ to be closed iff there exists an ideal I in the ring of polynomials over $\mathbb{k}$ of n variables such that $F = V(I)$, where $V(I) := \{\mathbf{x} \in \mathbb{k}^n \mid \forall f \in I : f(\mathbf{x}) = 0\}$. Show that in this way we obtain a topology on $\mathbb{k}^n$. This topology is called the *Zariski topology*.

3 Continuous Maps

Exercise 8. Suppose that X and Y are topological spaces. Prove that if X is discrete, then each mapping $f : X \to Y$ is continuous. Also prove that if Y is indiscrete, then each mapping $f : X \to Y$ is continuous.

Exercise 9. Let $f : X \to Y$ be a continuous map. Show that the preimage of a closed set in Y is closed in X.

Exercise 10. Show that a composition of continuous maps is continuous.

Exercise 11. Prove that $\mathbb{Z}$ and $\mathbb{Q}$ aren't homeomorphic. Both topological spaces are viewed as subspaces of $\mathbb{R}$.

Exercise 12. A mapping $f : X \to Y$ between topological spaces is called *continuous at a point* $x \in X$ if for each neighbourhood N of the point $f(x)$ its preimage $f^{-1}(N)$ is a neighbourhood of x. Show that a mapping $f : X \to Y$ between topological spaces is continuous iff it is continuous at each point of X.

[3] A *directed set* in a poset is a non-empty subset such that each pair of elements of the subset has an upper bound in the subset.

[4] A field is called *algebraically closed* if each non-constant polynomial with coefficients from this field has a root in this field.