Exercise 6) of $1/\lambda$ is

$$\hat{\Lambda}^{-1} = \frac{\pi}{n} \sum_{i=1}^{n} X_i^2.$$

An estimate of λ is thus made and hence, if the total area A is known, the total number of trees may be estimated as λA. For further details see Diggle (1975, 1983), Ripley (1981) and Upton and Fingleton (1985).

Method 2–Counting

Another method of testing the hypothesis of a Poisson forest is to subdivide the area of interest into N equal smaller areas called cells. The numbers N_k of cells containing k plants can be compared using a χ^2-test with the expected numbers under the Poisson assumption using (3.10), with $\bar{n}$ = the mean number of plants per cell.

Extensions to three and four dimensions

Suppose objects are randomly distributed throughout a 3-dimensional region. The above concepts may be extended by defining a Poisson point process in $\mathbb{R}^3$. Here, if A is a subset of $\mathbb{R}^3$, the number of objects in A is a Poisson random variable with parameter $\lambda|A|$, where λ is the mean number of objects per unit volume and $|A|$ is the volume of A. Such a point process will be useful in describing distributions of organisms in the ocean or the earth's atmosphere, distributions of certain rocks in the earth's crust and of objects in space. Similarly, a Poisson point process may be defined on subsets of $\mathbb{R}^4$ with a view to describing random events in space–time.

3.7 COMPOUND POISSON RANDOM VARIABLES

Let $X_k, k = 1, 2, \ldots$ be independent identically distributed random variables and let N be a non-negative integer-valued random variable, independent of the X_k. Then we may form the following sum:

$$S_N = X_1 + X_2 + \cdots + X_N, \tag{3.12}$$

where the number of terms is determined by the value of N. Thus S_N is a **random sum of random variables**: we take S_N to be zero if $N = 0$. If N is a Poisson random variable, S_N is called a **compound Poisson random variable**. The mean and variance of S_N are then as follows.

Theorem 3.8 **Let** $E(X_1) = \mu$ **and** $\mathrm{Var}(X_1) = \sigma^2$, $|\mu| < \infty, \sigma < \infty$. **If** N **is**

Poisson with parameter λ, then S_N defined by (3.12) has mean and variance

$$E(S_N) = \lambda\mu$$
$$\text{Var}(S_N) = \lambda(\mu^2 + \sigma^2).$$

Proof The law of total probability applied to expectations (see p. 8) gives

$$E(S_N) = \sum_{k=0}^{\infty} E(S_N | N = k) \Pr\{N = k\}.$$

But conditioned on $N = k$, there are k terms in (3.12) so $E(S_N | N = k) = k\mu$. Thus

$$\begin{aligned} E(S_N) &= \sum_{k=0}^{\infty} \mu k \Pr\{N = k\} \\ &= \mu E(N) \\ &= \lambda\mu. \end{aligned}$$

Similarly,

$$\begin{aligned} E(S_N^2) &= \sum_{k=0}^{\infty} E(S_N^2 | N = k) \Pr\{N = k\} \\ &= \sum_{k=0}^{\infty} [\text{Var}(S_N | N = k) + E^2(S_N | N = k)] \Pr\{N = k\} \\ &= \sum_{k=0}^{\infty} (k\sigma^2 + k^2\mu^2) \Pr\{N = k\} \\ &= \sigma^2 E(N) + \mu^2 E(N^2) \\ &= \sigma^2\lambda + \mu^2[\text{Var}(N) + E^2(N)] \\ &= \sigma^2\lambda + \mu^2(\lambda + \lambda^2). \end{aligned}$$

The result follows since $\text{Var}(S_N) = E(S_N^2) - \lambda^2\mu^2$.

Example The number of seeds (N) produced by a certain kind of plant has a Poisson distribution with parameter λ. Each seed, independently of how many there are, has probability p of forming into a developed plant. Find the mean and variance of the number of developed plants (ignoring the parent).

Solution Let $X_k = 1$ if the kth seed develops into a plant and let $X_k = 0$ if it doesn't. Then the X_k are i.i.d. Bernoulli random variables with

$$\Pr\{X_1 = 1\} = p = 1 - \Pr\{X_1 = 0\}$$

and

$$E(X_1) = p$$
$$\text{Var}(X_1) = p(1 - p).$$

The number of developed plants is

$$S_N = X_1 + X_2 + \cdots + X_N$$

which is therefore a compound Poisson random variable. By Theorem 3.8, with $\mu = p$ and $\sigma^2 = p(1-p)$ we find

$$\begin{aligned} E(S_N) &= \lambda p \\ \mathrm{Var}(S_N) &= \lambda(p^2 + p(1-p)) \\ &= \lambda p. \end{aligned}$$

As might be suspected from these results, in this example S_N is itself a Poisson random variable with parameter λp. This can be readily shown using generating functions – see Section 10.4.

3.8 THE DELTA FUNCTION

We will consider an interesting neurophysiological application of compound Poisson random variables in the next section. Before doing so we find it convenient to introduce the delta function. This was first employed in quantum mechanics by the celebrated theoretical physicist P.A.M. Dirac, but has since found application in many areas.

Let X_ε be a random variable which is uniformly distributed on $(x_0 - \varepsilon/2, x_0 + \varepsilon/2)$. Then its distribution function is

$$F_{X_\varepsilon}(x) = \Pr\{X_\varepsilon \leqslant x\} = \begin{cases} 0, & x \leqslant x_0 - \varepsilon/2, \\ \dfrac{1}{\varepsilon}[x - (x_0 - \varepsilon/2)], & |x - x_0| < \varepsilon/2, \\ 1, & x \geqslant x_0 + \varepsilon/2 \end{cases}$$
$$\doteq H_\varepsilon(x - x_0).$$

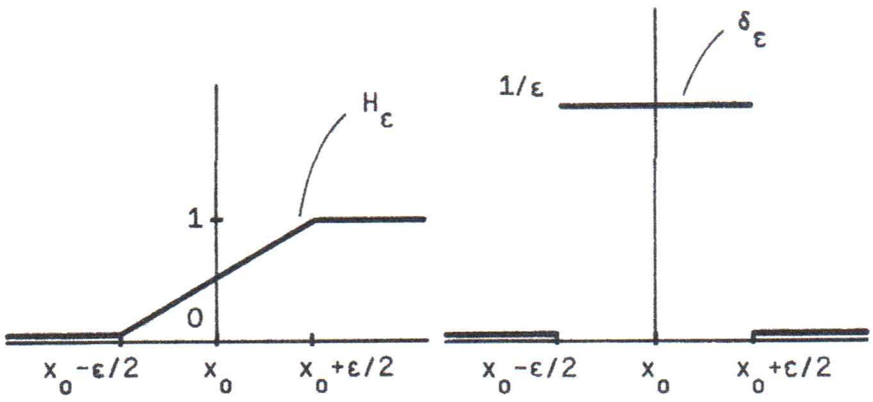

Figure 3.8

The density of X_ε is

$$f_{X_\varepsilon}(x) = \frac{\mathrm{d}F_{X_\varepsilon}}{\mathrm{d}x} = \begin{cases} 1/\varepsilon, & |x - x_0| < \varepsilon/2, \\ 0, & \text{otherwise.} \end{cases}$$

$$\doteq \delta_\varepsilon(x - x_0).$$

The functions H_ε and δ_ε are sketched in Fig. 3.8.

As $\varepsilon \to 0$, $H_\varepsilon(x - x_0)$ approaches the unit step function, $H(x - x_0)$ and $\delta_\varepsilon(x - x_0)$ approaches what is called a delta function, $\delta(x - x_0)$. In the limit as $\varepsilon \to 0$, δ_ε becomes 'infinitely large on an infinitesimally small interval' and zero everywhere else. We always have for all $\varepsilon > 0$,

$$\int_{-\infty}^{\infty} \delta_\varepsilon(x - x_0)\,\mathrm{d}x = 1.$$

We say that the limiting object $\delta(x - x_0)$ is a **delta function** or a **unit mass** concentrated at x_0.

Substitution property

Let f be an arbitrary function which is continuous on $(x_0 - \varepsilon/2, x_0 + \varepsilon/2)$. Consider the integrals

$$I_\varepsilon = \int_{-\infty}^{\infty} f(x)\delta_\varepsilon(x - x_0)\,\mathrm{d}x = \frac{1}{\varepsilon}\int_{x_0 - \varepsilon/2}^{x_0 + \varepsilon/2} f(x)\,\mathrm{d}x.$$

When ε is very small,

$$I_\varepsilon \simeq \frac{1}{\varepsilon}\varepsilon f(x_0) = f(x_0).$$

We thus obtain the **substitution property** of the delta function:

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0). \tag{3.13}$$

Technically this relation is used to define the delta function in the theory of **generalized functions** (see for example Griffel, 1985). With $f(x) = 1$, (3.13) becomes

$$\int_{-\infty}^{\infty} \delta(x - x_0)\,\mathrm{d}x = 1.$$

Furthermore, since $\delta(x) = 0$ for $x \neq 0$,

$$\int_{-\infty}^{x} \delta(x' - x_0)\,\mathrm{d}x' = H(x - x_0) = \begin{cases} 0, & x < x_0, \\ 1, & x \geqslant x_0. \end{cases}$$

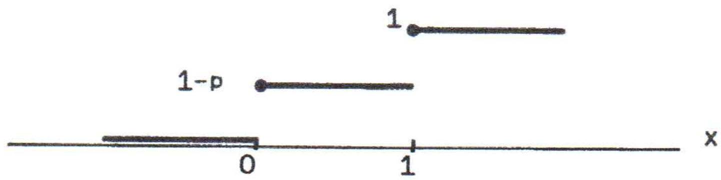

Figure 3.9

Thus we may informally regard $\delta(x - x_0)$ as the derivative of the unit step function $H(x - x_0)$. Thus it may be viewed as the density of the constant x_0.

Probability density of discrete random variables

Let X be a discrete random variable with $\Pr(X = 1) = 1 - \Pr(X = 0) = p$. Then the probability density of X is written

$$f_X(x) = (1 - p)\delta(x) + p\delta(x - 1).$$

This gives the correct distribution function for X because

$$F_X(x) = \Pr(X \leqslant x) = \int_{-\infty}^{x} f_X(x')\,dx' = (1 - p)H(x) + pH(x - 1)$$

$$= \begin{cases} 0, & x < 0, \\ 1 - p, & 0 \leqslant x < 1, \\ 1, & x \geqslant 1, \end{cases}$$

as is sketched in Fig. 3.9.

Similarly, the probability density of a Poisson random variable with parameter λ is given by

$$f_X(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta(x - k).$$

3.9 AN APPLICATION IN NEUROBIOLOGY

In Section 3.5 we mentioned the small voltage changes which occur spontaneously at nerve–muscle junctions. Their **arrival times** were found to be well described by a Poisson point process in time. Here we are concerned with their **magnitudes**. Figure 3.10 depicts the anatomical arrangement at the nerve–muscle junction. Each cross represents a potentially active site.

The small **spontaneous** voltage changes have amplitudes whose histogram is fitted to a normal density – see Fig. 3.11. When a **nerve impulse**, having travelled out from the spinal cord, enters the junction it elicits a much bigger

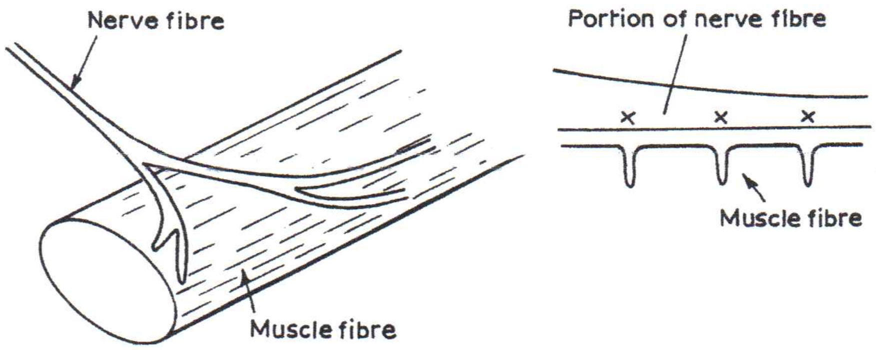

Figure 3.10 The arrangement at a nerve–muscle junction.

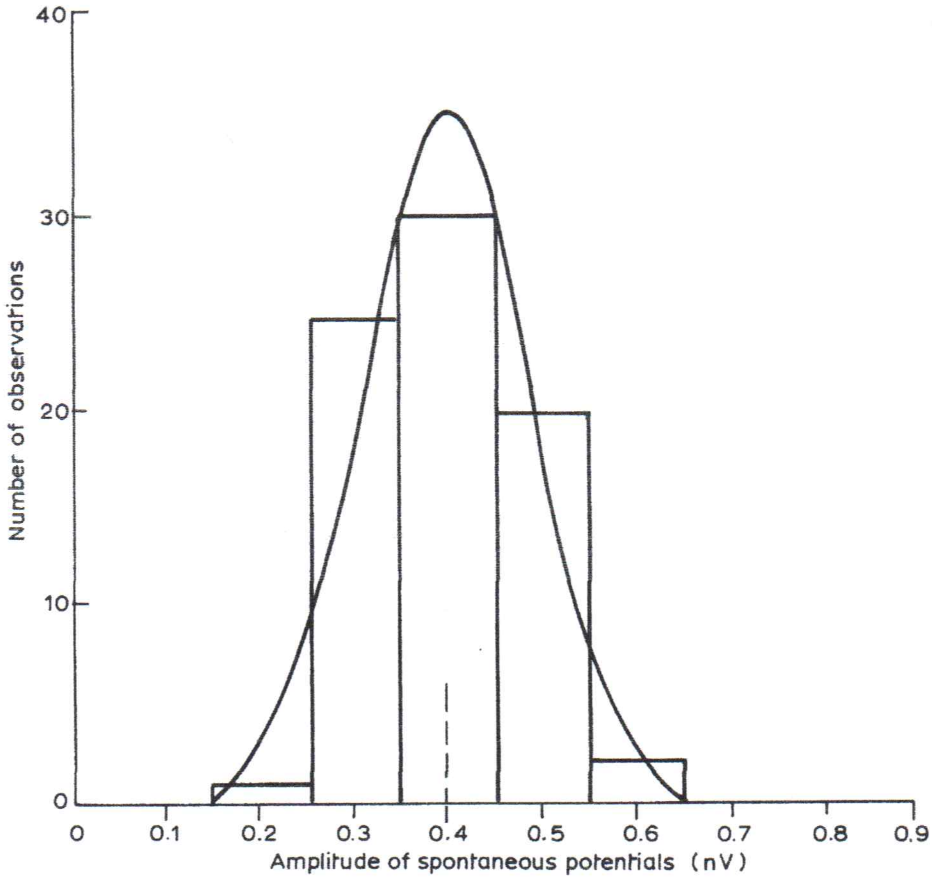

Figure 3.11 Histogram of small spontaneous voltage changes and fitted normal density. From Martin (1977). Figures 3.11–3.13 reproduced with permission of the American Physiological Society and the author.

response whose amplitude we will call V. It was hypothesized that the large response was composed of many unit responses, the latter corresponding to the spontaneous activity.

We assume that the unit responses are $X_1, X_2, \ldots$ and that these are normal with mean μ and variance σ^2. A large response consists of a random number N of the unit responses. If $N = 0$, there is no response at all. Thus

$$V = X_1 + X_2 + \cdots + X_N,$$

which is a random sum of random variables. A natural choice for N is a binomial random variable with parameters n and p where n is the number of potentially active sites and p is the probability that any site is activated. However, the assumption is usually made that N is Poisson. This is based on the Poisson approximation to the binomial and the fact that a Poisson distribution is characterized by a single parameter. Hence V is a compound Poisson random variable. The probability density of V is then found as follows:

$$\Pr\{V \in (v, v + \mathrm{d}v)\} = \sum_{k=0}^{\infty} \Pr\{V \in (v, v + \mathrm{d}v) | N = k\} \Pr\{N = k\}$$

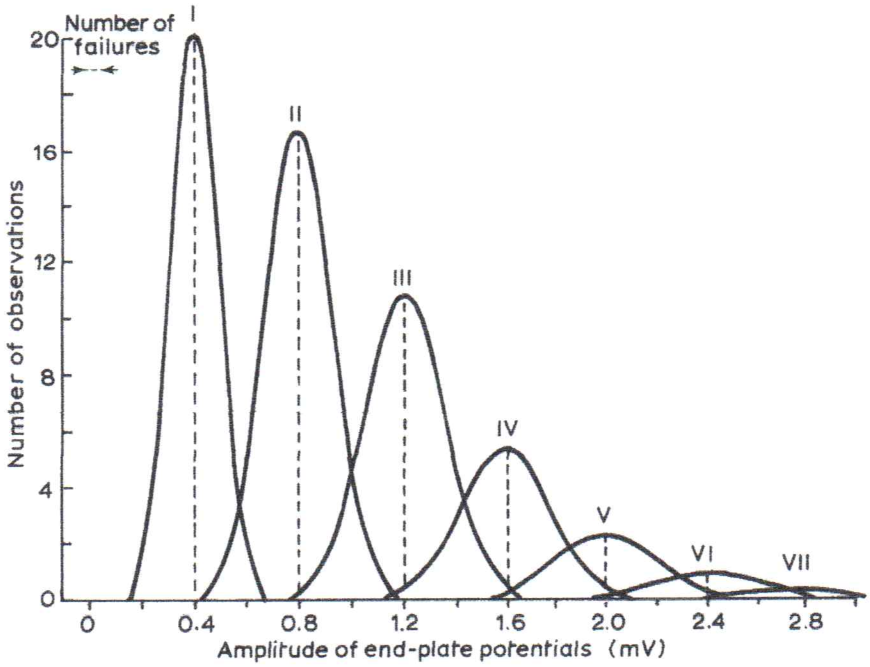

Figure 3.12 Decomposition of the compound Poisson distribution. The curve marked I corresponds to p_1, the curve marked II to p_2, etc., in (3.14).

$$= e^{-\lambda} \Pr\{V \in (v, v + dv) | N = 0\}$$
$$+ \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \Pr\{V \in (v, v + dv) | N = k\}$$
$$= \left[e^{-\lambda}\delta(v) + e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \frac{1}{\sqrt{2\pi k\sigma^2}} \exp\left(\frac{-(v - k\mu)^2}{2k\sigma^2} \right) \right] dv$$
$$\doteq \left[e^{-\lambda}\delta(v) + \sum_{k=1}^{\infty} p_k(v) \right] dv,$$

where $\delta(v)$ is a delta function concentrated at the origin. Hence the required density is

$$\boxed{f_V(v) = e^{-\lambda}\delta(v) + \sum_{k=1}^{\infty} p_k(v)}. \tag{3.14}$$

The terms in the expansion of the density of V are shown in Fig. 3.12. The density of V is shown in Fig. 3.13 along with the empirical distribution. Excellent agreement is found between theory and experiment, providing a validation of the 'quantum hypothesis'. For further details see Martin (1977).

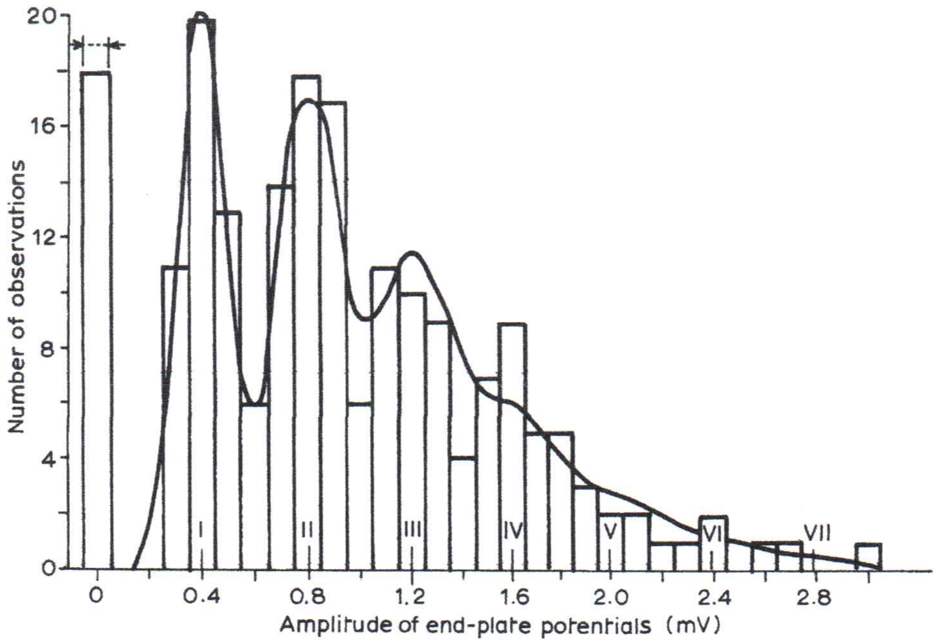

Figure 3.13 Histogram of responses. The curve is the density for the compound Poisson distribution, the column at 0 corresponding to the delta function in (3.14).