After $n = 10\,000$ steps with $p = 0.6$, $E(X_n) = 2000$ an

$$\Pr\{1808 < X_{10\,000} < 2192\} \simeq 0.95,$$

whereas when $p = 0.5$ the mean is 0 and

$$\Pr\{-196 < X_{10\,000} < 196\} \simeq 0.95.$$

Figure 7.3 shows the growth of the mean with increasing n and the approximating normal densities at $n = 50$ and $n = 100$ for various p.

7.3 RANDOM WALK WITH ABSORBING STATES

The paths of the process considered in the previous section increase or decrease at random, indefinitely. In many important applications this is not the case as particular values have special significance. This is illustrated in the following classical example.

A simple gambling game

Let two gamblers, A and B, initially have \$$a$ and \$$b$, respectively, where a and b are positive integers. Suppose that at each round of their game, player A wins \$1 from B with probability p and loses \$1 to B with probability $q = 1 - p$. The total capital of the two players at all times is

$$c = a + b.$$

Let X_n be player A's capital at round n where $n = 0, 1, 2, \ldots$ and $X_0 = a$. Let Z_n be the amount A wins on trial n. The Z_n are assumed to be independent.

It is clear that as long as both players have money left,

$$X_n = X_{n-1} + Z_n, \qquad n = 1, 2, \ldots,$$

where the Z_n are i.i.d. as in the previous section. Thus $\{X_n, n = 0, 1, 2, \ldots\}$ is a simple random walk but there are now some restrictions or boundary conditions on the values it takes.

Absorbing states

Let us assume that A and B play until one of them has no money left; i.e., has 'gone broke'. This may occur in two ways. A's capital may reach zero or A's capital may reach c, in which case B has gone broke. The process $X = \{X_0, X_1, X_2, \ldots\}$ is thus restricted to the set of integers $\{0, 1, 2, \ldots, c\}$ and it terminates when either the value 0 or c is attained. The values 0 and c are called absorbing states, or we say there are **absorbing barriers** at 0 and c. Figure 7.4 shows plots of A's capital X_n versus trial number for two possible

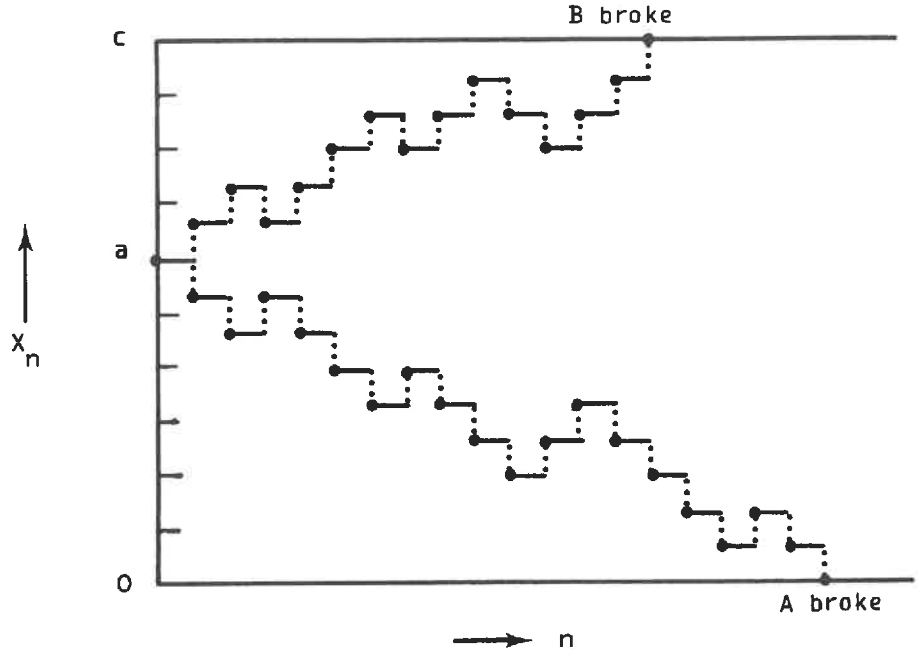

Figure 7.4 Two sample paths of a simple random walk with absorbing barriers at 0 and c. The upper path results in absorption at c (corresponding to player A winning all the money) and the lower one in absorption at 0 (player A broke).

games. One of these sample paths leads to absorption of X at 0 and the other to absorption at c.

7.4 THE PROBABILITIES OF ABSORPTION AT 0

Let P_a, $a = 0, 1, 2, \ldots, c$ denote the probabilities that player A goes broke when his initial capital is \$$a$. Equivalently P_a is the probability that X is absorbed at 0 when $X_0 = a$. The calculation of P_a is referred to as a **gambler's ruin problem**. We will obtain a difference equation for P_a.

First, however, we observe that the following boundary conditions must apply:

$$\boxed{\begin{aligned} P_0 &= 1 \\ P_c &= 0 \end{aligned}}$$

since if $a = 0$ the probability of absorption at 0 is one whereas if $a = c$, absorption at c has already occurred and absorption at 0 is impossible.

Now, when a is not equal to either 0 or c, all games can be divided into two mutually exclusive categories:

(i) A wins the first round;
(ii) A loses the first round.

Thus the event $\{A \text{ goes broke from } a\}$ is the union of two mutually exclusive events:

$$\begin{aligned}&\{A \text{ goes broke from } a\} = \\ &\{A \text{ wins the first round and goes broke from } a+1\} \\ &\quad \cup \{A \text{ loses the first round and goes broke from } a-1\}. \qquad (7.6)\end{aligned}$$

Also, since going broke after winning the first round and winning the first round are independent,

$$\begin{aligned}&\Pr\{A \text{ wins the first round and goes broke from } a+1\} \\ &= \Pr\{A \text{ wins the first round}\} \Pr\{A \text{ goes broke from } a+1\} \\ &= pP_{a+1}. \qquad (7.7)\end{aligned}$$

Similarly,

$$\begin{aligned}&\Pr\{A \text{ loses the first round and goes broke from } a-1\} \\ &\quad = qP_{a-1}. \qquad (7.8)\end{aligned}$$

Since the probability of the union of two mutually exclusive events is the sum of their individual probabilities, we obtain from (7.6)–(7.8), the key relation

$$\boxed{P_a = pP_{a+1} + qP_{a-1}}, \qquad a = 1, 2, \ldots, c-1. \qquad (7.9)$$

This is a difference equation for P_a which we will solve subject to the above boundary conditions.

Solution of the difference equation (7.9)

There are three main steps in solving (7.9).

(i) *The first step is to rearrange the equation*

Since $p + q = 1$, we have

$$(p+q)P_a = pP_{a+1} + qP_{a-1},$$

or

$$p(P_{a+1} - P_a) = q(P_a - P_{a-1}).$$

Dividing by p and letting

$$r = \frac{q}{p}$$

gives

$$P_{a+1} - P_a = r(P_a - P_{a-1}).$$

(ii) *The second step is to find P_1.*

To do this we write out the system of equations and utilize the boundary condition $P_0 = 1$:

$$\left.\begin{array}{llll} a=1 & : P_2 - P_1 & = r(P_1 - P_0) & = r(P_1 - 1) \\ a=2 & : P_3 - P_2 & = r(P_2 - P_1) & = r^2(P_1 - 1) \\ \vdots & \quad\vdots & \quad\vdots & \quad\vdots \\ a=c-2 & : P_{c-1} - P_{c-2} & = r(P_{c-2} - P_{c-3}) & = r^{c-2}(P_1 - 1) \\ a=c-1 & : P_c - P_{c-1} & = r(P_{c-1} - P_{c-2}) & = r^{c-1}(P_1 - 1) \end{array}\right\} \qquad (7.10)$$

Adding all these and cancelling gives

$$P_c - P_1 = -P_1 = (P_1 - 1)(r + r^2 + \cdots + r^{c-1}), \qquad (7.11)$$

where we have used the fact that $P_c = 0$.

Special case: $p = q = \frac{1}{2}$ If $p = q = \frac{1}{2}$ then $r = 1$ so $r + r^2 + \cdots + r^{c-1} = c - 1$. Hence

$$-P_1 = (P_1 - 1)(c - 1).$$

Solving this gives

$$\boxed{P_1 = 1 - \frac{1}{c}}, \qquad r = 1. \qquad (7.12)$$

General case: $p \neq q$ Equation (7.11) can be rearranged to give

$$(P_1 - 1)(1 + r + r^2 + \cdots + r^{c-1}) + 1 = 0$$

so

$$P_1 = 1 - \frac{1}{1 + r + r^2 + \cdots + r^{c-1}}.$$

For $r \neq 1$ we utilize the following formula for the sum of a finite number of terms of a geometric series:

$$1 + r + r^2 + \cdots + r^{c-1} = \frac{1 - r^c}{1 - r}. \qquad (7.13)$$

Hence, after a little algebra,

$$\boxed{P_1 = \frac{r - r^c}{1 - r^c}}, \qquad r \neq 1. \qquad (7.14)$$

Equations (7.12) and (7.14) give the probabilities that the random walk is absorbed at zero when $X_0 = 1$, or the chances that player A goes broke when starting with one unit of capital.

(iii) *The third and final step is to solve for* P_a, $a \neq 1$.
From the system of equations (7.10) we get

$$\begin{aligned} P_2 &= P_1 + r(P_1 - 1) \\ P_3 &= P_2 + r^2(P_1 - 1) \quad = P_1 + (P_1 - 1)(r + r^2) \\ &\vdots \\ P_a &= P_{a-1} + r^{a-1}(P_1 - 1) = P_1 + (P_1 - 1)(r + r^2 + \cdots + r^{a-1}). \end{aligned}$$

Adding and subtracting one gives

$$P_a = (P_1 - 1)(1 + r + r^2 + \cdots + r^{a-1}) + 1. \tag{7.15}$$

Special case: $p = q = \frac{1}{2}$ When $r = 1$ we have $1 + r + r^2 + \cdots + r^{a-1} = a$, so using (7.12) gives

$$\boxed{P_a = 1 - \frac{a}{c}}, \qquad p = q. \tag{7.16}$$

General case: $p \neq q$ From (7.14) we find

$$P_1 - 1 = \frac{r-1}{1-r^c}.$$

Substituting this in (7.15) and utilizing (7.13) for the sum of the geometric series,

$$P_a = \left(\frac{r-1}{1-r^c}\right)\left(\frac{1-r^a}{1-r}\right) + 1,$$

which rearranges to

$$\boxed{P_a = \frac{r^a - r^c}{1 - r^c}}, \qquad r \neq 1.$$

Thus, in terms of p and q we finally obtain the following results.

Theorem 7.1 The probability that the random walk is absorbed at 0 when it starts at $X_0 = a$, (or the chances that player A goes broke from a) is

$$\boxed{P_a = \frac{(q/p)^a - (q/p)^c}{1 - (q/p)^c}}, \qquad p \neq q. \tag{7.17}$$

Table 7.1 Values of P_a for various values of p.

a	$p=0.25$	$p=0.4$	$p=0.5$
0	1	1	1
1	0.99997	0.99118	0.9
2	0.99986	0.97794	0.8
3	0.99956	0.95809	0.7
4	0.99865	0.92831	0.6
5	0.99590	0.88364	0.5
6	0.98767	0.81663	0.4
7	0.96298	0.71612	0.3
8	0.88890	0.56536	0.2
9	0.66667	0.33922	0.1
10	0	0	0

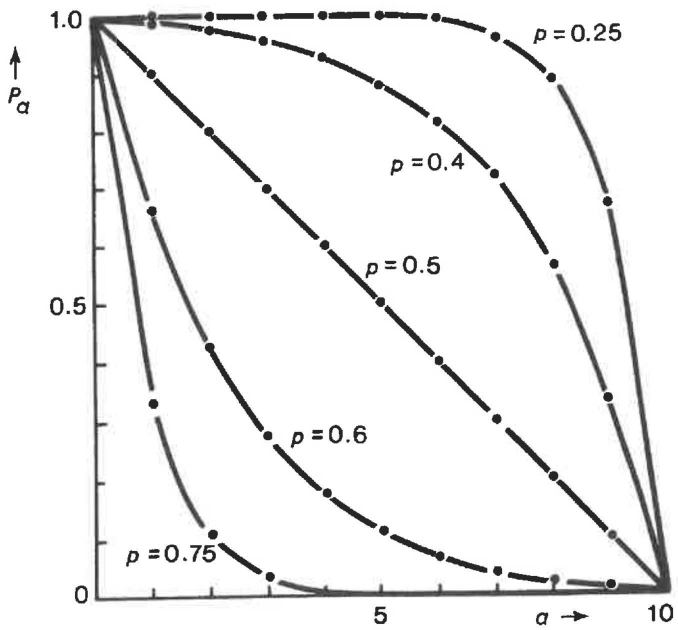

Figure 7.5 The probabilities P_a that player A goes broke. The total capital of both players is 10, a is the initial capital of A, and p = chance that A wins each round.

When $p=q=\frac{1}{2}$,

$$\boxed{P_a = 1 - \frac{a}{c}}$$

Some numerical values

Table 7.1 lists values of P_a for $c=10$, $a=0,1,\ldots,10$ for the three values $p=0.25$, $p=0.4$ and $p=0.5$. The values of P_a are plotted against a in Fig. 7.5. Also shown are curves for $p=0.75$ and $p=0.6$ which are obtained from the relation (see Exercise 8)

$$P_a(p) = 1 - P_{c-a}(1-p).$$

In the case shown where $p=0.25$, the chances are close to one that X will be absorbed at 0 (A will go broke) unless X_0 is 8 or more. Clearly the chances that A does not go broke are promoted by:

(i) a large p value, i.e. a high probability of winning each round;
(ii) a large value of X_0, i.e. a large share of the initial capital.

7.5 ABSORPTION AT $c>0$

We have just considered the random walk $\{X_n, n=0,1,2,\ldots\}$ where X_n was player A's fortune at epoch n. Let Y_n be player B's fortune at epoch n. Then $\{Y_n, n=0,1,2,\ldots\}$ is a random walk with probability q of a step up and p of a step down at each time unit. Also, $Y_0=c-a$ and if Y is absorbed at 0 then X is absorbed at c.

The quantity

$$Q_a = \Pr\{X \text{ is absorbed at } c \text{ when } X_0=a\},$$

can therefore be obtained from the formulas for P_a by replacing a by $c-a$ and interchanging p and q.

Special case: $p=q=\frac{1}{2}$ In this case $P_a=1-a/c$ so $Q_a=1-(c-a)/c$. Hence

$$\boxed{Q_a = \frac{a}{c}}, \qquad p=q.$$

General case: $p\neq q$ From (7.17) we obtain

$$Q_a = \frac{(p/q)^{c-a}-(p/q)^c}{1-(p/q)^c}.$$

Multiplying the numerator and denominator by $(q/p)^c$ and rearranging gives

$$\boxed{Q_a = \frac{1-(q/p)^a}{1-(q/p)^c}}, \qquad p \neq q.$$

In all cases we find

$$\boxed{P_a + Q_a = 1} \tag{7.18}$$

Thus absorption at one or the other of the absorbing states is a certain event.

That the probabilities of absorption at 0 and at c add to unity is not obvious. One can imagine that a game might last forever, with A winning one round, B winning the next, A the next, and so on. Equation (7.18) tells us that the probability associated with such never-ending sample paths is zero. Hence sooner or later the random walk is absorbed, or in the gambling context, one of the players goes broke.

7.6 THE CASE $c = \infty$

If a, which is player A's initial capital, is kept finite and we let b become infinite, then player A is gambling against an opponent with infinite capital. Then, since $c = a + b$, c becomes infinite. The chances that player A goes broke are obtained by taking the limit $c \to \infty$ in expressions (7.16) and (7.17) for P_a. There are three cases to consider.

(i) $p > q$

Then player A has the advantage and since $q/p < 1$,

$$\lim_{c\to\infty} P_a = \lim_{c\to\infty} \frac{(q/p)^a - (q/p)^c}{1-(q/p)^c} = (q/p)^a,$$

which is less than one.

(ii) $p = q$

Then the game is 'fair' and

$$\lim_{c\to\infty} P_a = \lim_{c\to\infty} 1 - \frac{a}{c} = 1.$$

(iii) $p < q$

Here player A is disadvantaged and

$$\lim_{c\to\infty} P_a = \lim_{c\to\infty} \frac{(q/p)^a - (q/p)^c}{1-(q/p)^c} = 1$$

since $q/p > 1$.

Note that even when A and B have equal chances to win each round, player A goes broke for sure when player B has infinite initial capital. In casinos the situation is approximately that of a gambler playing someone with infinite capital, and, to make matters worse $p<q$ so the gambler goes broke with probability one if he keeps on playing. Casino owners are not usually referred to as gamblers!

7.7 HOW LONG WILL ABSORPTION TAKE?

In Section 7.5 we saw that the random walk X on a finite interval is certain to be absorbed at 0 or c. We now ask how long this will take.

Define the random variable

$$T_a = \text{time to absorption of } X \text{ when } X_0 = a, \qquad a = 0, 1, 2, \ldots, c.$$

The probability distribution of T_a can be found exactly (see for example Feller, 1968, Chapter 14) but we will find only the expected value of T_a:

$$D_a = E(T_a).$$

Clearly, if $a=0$ or $a=c$, then absorption is immediate so we have the boundary conditions

$$\boxed{D_0 = 0} \qquad (7.19)$$

$$\boxed{D_c = 0} \qquad (7.20)$$

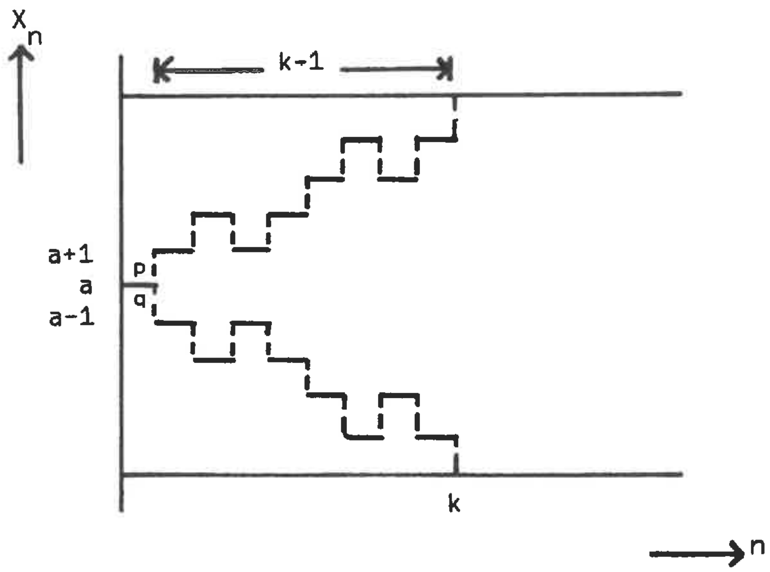

Figure 7.6 Paths leading to absorption after k steps.

We will derive a difference equation for D_a. Define

$$P(a,k) = \Pr\{T_a = k\},\ k = 1,2,\ldots$$

which is the probability that absorption takes k time units when the process begins at a. Considering the possible results of the first round as before (see the sketch in Fig. 7.6), we find

$$P(a,k) = pP(a+1,k-1) + qP(a-1,k-1).$$

Multiplying by k and summing over k gives

$$E(T_a) = \sum_{k=1}^{\infty} kP(a,k) = p\sum_{k=1}^{\infty} kP(a+1,k-1) + q\sum_{k=1}^{\infty} kP(a-1,k-1).$$

Putting $j = k-1$ this may be rewritten

$$\begin{aligned} D_a &= p\sum_{j=0}^{\infty} (j+1)P(a+1,j) + q\sum_{j=0}^{\infty} (j+1)P(a-1,j) \\ &= p\sum_{j=0}^{\infty} jP(a+1,j) + q\sum_{j=0}^{\infty} jP(a-1,j) \\ &\quad + p\sum_{j=0}^{\infty} P(a+1,j) + q\sum_{j=0}^{\infty} P(a-1,j). \end{aligned}$$

But we have seen that absorption is certain, so

$$\sum_{j=0}^{\infty} P(a+1,j) = \sum_{j=0}^{\infty} P(a-1,j) = 1.$$

Hence

$$D_a = pD_{a+1} + qD_{a-1} + p + q$$

Table 7.2 Values of D_a from (7.22) and (7.23) with $c = 10$

a	$p = 0.25$	$p = 0.4$	$p = 0.5$
0	0	0	0
1	1.999	4.559	9
2	3.997	8.897	16
3	5.991	12.904	21
4	7.973	16.415	24
5	9.918	19.182	25
6	11.753	20.832	24
7	13.260	20.806	21
8	13.778	18.268	16
9	11.334	11.961	9
10	0	0	0

or, finally,

$$\boxed{D_a = pD_{a+1} + qD_{a-1} + 1}, \qquad a = 1, 2, \ldots, c-1. \tag{7.21}$$

This is the desired difference equation for D_a, which can be written down without the preceding steps (see Exercise 11).

The solution of (7.21) may be found in the same way that we solved the difference equation for P_a. In Exercise 12 it is found that the solution satisfying the boundary conditions (7.19), (7.20) is

$$\boxed{D_a = a(c-a)}, \qquad p = q, \tag{7.22}$$

$$\boxed{D_a = \frac{1}{q-p}\left(a - c\left\{\frac{1-(q/p)^a}{1-(q/p)^c}\right\}\right)}, \qquad p \neq q. \tag{7.23}$$

Numerical values

Table 7.2 lists calculated expected times to absorption for various values of a when $c = 10$ and for $p = 0.25$, $p = 0.4$ and $p = 0.5$. These values are plotted as functions of a in Fig. 7.7.

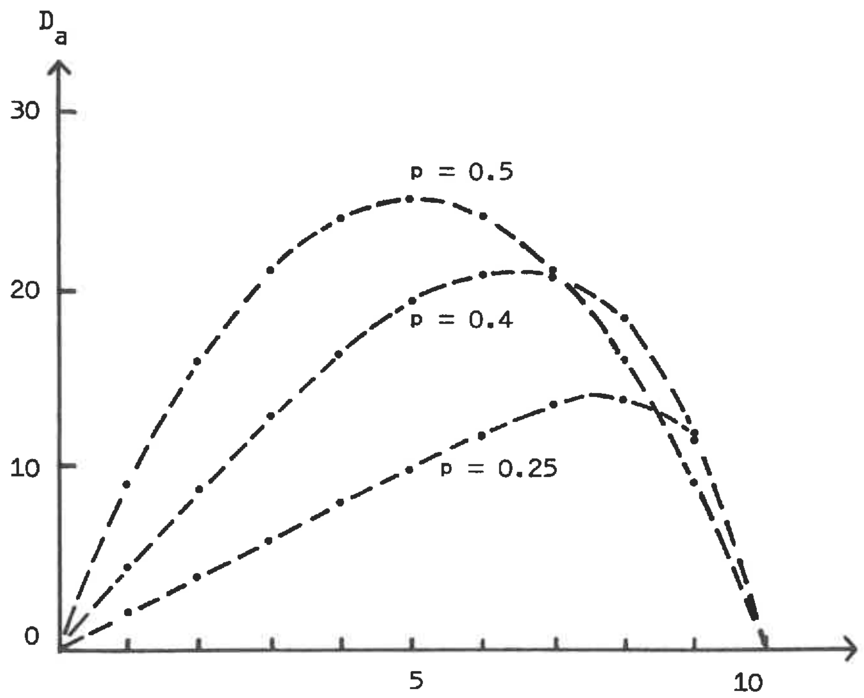

Figure 7.7 The expected times to absorption, D_a, of the simple random walk starting at a when $c = 10$ for various p.