Mistake in inverse for trivialization of S^n:

$$\phi^{-1}(z, \lambda x) = \sqrt{\lambda+1}\left(z + \sqrt{(1-\langle z,z\rangle)}\, x\right)$$

We have a notion of submanifolds of manifolds:

<u>Def 2.21</u> Suppose $(M, \mathcal{A})$ manifold of dim. n and let $k \leq n$. A subset $N \subseteq M$ is a submanifold of M of dim. k, if for any $x \in N$ and any chart $(U, u) \in \mathcal{A}$ with $x \in U$, the subset $u(U \cap N) \subseteq \mathbb{R}^n$ is a k-dim. submfd. in $\mathbb{R}^n$.

(It's enough to ask for $\exists$ of one such chart).

Given N submfd of $(M, \mathcal{A})$, then for any $x \in N$ we can find a chart $(U, u) \in \mathcal{A}$ s.t. $u|_{U \cap N}$ has values in $\mathbb{R}^k = \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$.

These charts $(U \cap N, u|_{U \cap N})$ form an atlas of N making N in a k-dim. manifold.
This is called the standard manifold structure of a submfd $N \subseteq M$.

For $M = \mathbb{R}^n$ then Def. 2.21 coincides with Def. 2.8.

The standard mfd. structure of $N \subseteq M$ has the following property:

Prop. 2.22 $(M, \mathcal{U})$ manifold and $N \subseteq M$ a submfd.
Then the inclusion $i : N \hookrightarrow M$ is a smooth injective immersion and the standard mfd.-structure on N is the unique one satisfying the following universal property: For any mfd. P a map $f : P \to N$ is smooth $\iff$ $i \circ f : P \to M$ smooth.

Proof

(a) Standard mfd. str. satisfies univ. property:

With respect to charts $i \circ f$ is of the form

$$\underset{\subseteq \mathbb{R}^m}{W} \xrightarrow{f} \underset{\subseteq \mathbb{R}^k}{W'} \xhookrightarrow{i} \mathbb{R}^n$$ is smooth $\iff$ $f : W \to W'$ is smooth.

Applied to $f = \mathrm{Id}_N$ implies that $i : N \hookrightarrow M$ is smooth.

(b) Remains to show that this property characterizes mfd. str. on N:

Suppose N is equipped with two diff. mfd.-structures:

$$(N, \mathcal{B}) \quad \text{and} \quad (N, \mathcal{B}').$$

$$\mathrm{Id} : (N, \mathcal{B}) \longrightarrow (N, \mathcal{B}') \text{ is smooth} \iff$$

$$i : (N, \mathcal{B}) \hookrightarrow (M, \mathcal{A}) \text{ and } i : (N, \mathcal{B}') \hookrightarrow (M, \mathcal{A})$$

are smooth, which is the case by (a) and the assumption that $(N, \mathcal{B})$ and $(N, \mathcal{B}')$ satisfy the univ. property.

Def. 2.23 M, N manifolds. Then a smooth map $f: M \to N$ is called a (smooth) embedding, if the following holds:

- $f: M \longrightarrow f(M)$ homeomorphism
- $f: M \longrightarrow N$ is an immersion.

Images of embeddings are submanifolds:

Prop. 2.24 M, N smooth mfds. Then $f: M \to N$ is an embedding $\Longleftrightarrow$

- $f(M) \subseteq N$ is a submfd. of N.
- $f|_M : M \to f(M)$ is a diffeom.

<u>Proof.</u>

"$\Leftarrow$" $f: M \to N$ is smooth immersion because it equals composition $M \xrightarrow{f} f(M) \xhookrightarrow{i} N$.

"$\Rightarrow$" Since $f: M \to f(M)$ is a homeom., it remains that $f(M) \subseteq N$ is a submfd. and $f: M \to f(M)$ a local diffeom.

Both properties are of local nature, so we only need to verify them in neighbhd. of any $x \in M$ and $f(x) \in N$.

Fix $x \in M$, a chart (U, u) with $x \in U$

a chart (V, v) with $f(x) \in V$.

Since f is homeom. onto $f(M)$, we may assume $f(U) \subseteq V \cap f(M)$.
Replace f by $v \circ f \circ u^{-1}$ reduces the statement to the case where M and N are open subset of $\mathbb{R}^m$ and $\mathbb{R}^n$ and the result follows from Thm. 2.5. ($\exists$ local parametrization).

Prop. 2.25 Suppose M, N mfds. of dim. n and m respectively and $f: M \to N$ smooth map of constant rank r. Then for any $y \in f(M)$, $f^{-1}(y) \subseteq M$ is a submanifold of dim $n-r$ in M.

Proof Exercise / Homework.

Remark View $M = \mathbb{R}$ as topolog. manifold.
Then $\mathcal{A}_1 = \{ \mathrm{Id}: \mathbb{R} \to \mathbb{R} \}$ and $\mathcal{A}_2 = \{ u(x) = x^3 : \mathbb{R} \to \mathbb{R} \}$ ~~are~~ give rise to smooth structures, but they are not compatible C^∞-atlases, since $u^{-1} \circ \mathrm{Id}: x \mapsto \sqrt[3]{x}$ ~~here~~ is not smooth. But they are diffeomorphic via $f = \sqrt[3]{x}$.

- Any topolog. mfd. of dim. ≤ 3 admits a unique C^ω-structure. If two C^ω-mfd. of dim ≤ 3 are homeom., they are C^ω-diffeom.
- $\exists$ topolog. mfds. without any C^1-structure and $\exists$ some with many different ones:

For S^n the isomorphism classes of C^∞-struct. are as follows:

n	≤ 3	4	5,6	7	8	9	10	11	12	13	14	15
# isom. classes	1	≥ 1	1	28	2	8	6	992	1	3	2	16256

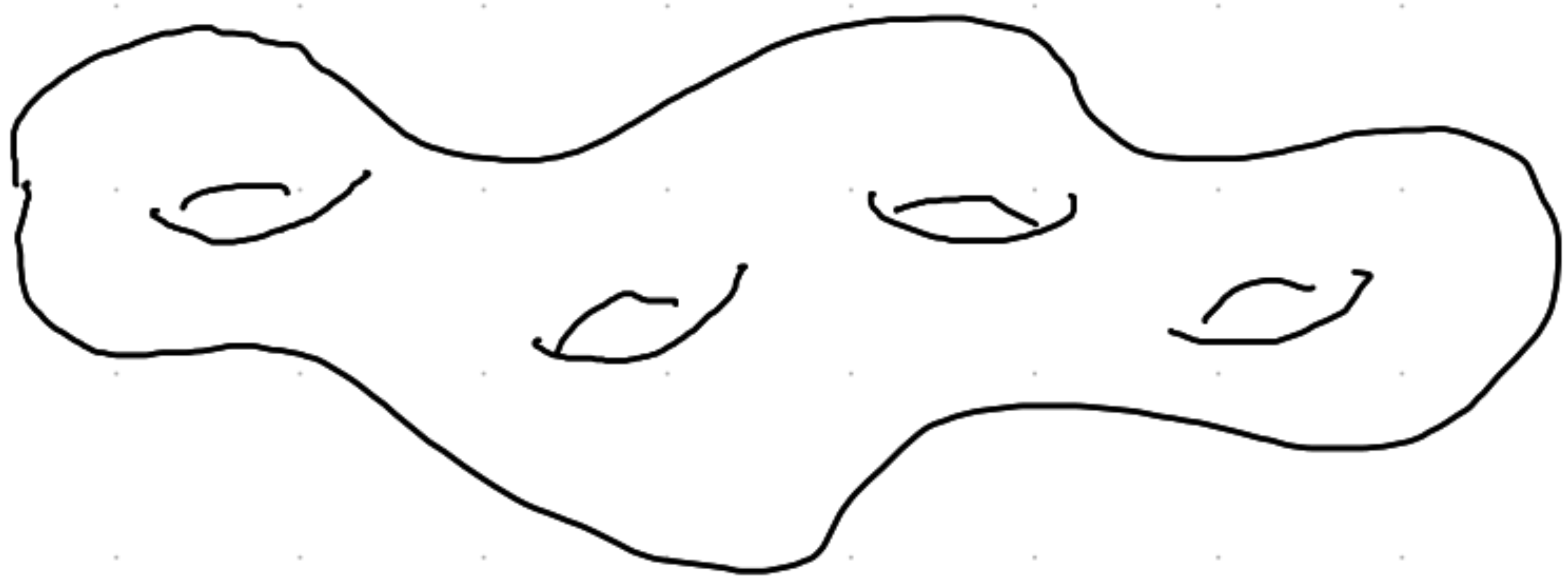
For dim ≥ 4 classification of topolog. and smooth mfds. differ.

For $n \neq 4$ $\mathbb{R}^n$ has a unique smooth structure, but $\mathbb{R}^4$ has uncountably many.

- The classification of topolog. mfds. of dim 1 and 2 is known:

  - Any 1-dim. connected mfd. is homeom. to $\mathbb{R}$ or S^1.

  - Any two dim. compact connected top. mfd. is homeom. to the connected sum of $g \geq 0$ copies of T^2 $\#^g T^2$, where $\#^0 T^2 := S^2$ or to the connected sum of $g \geq 1$ copies of $\mathbb{R}P^2$.
    Any of these are not homeomorphic.

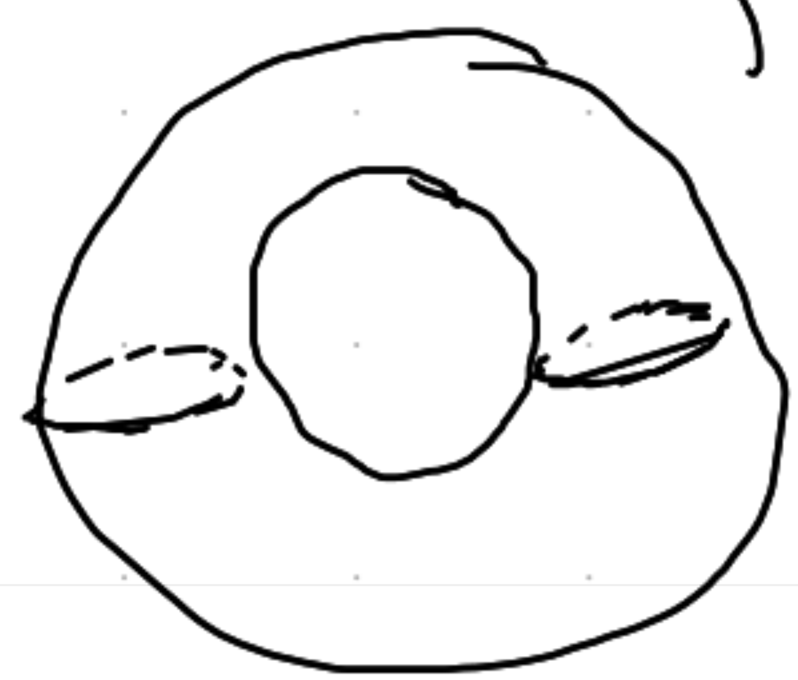


$\#^4 T^2$

g = genus of M.

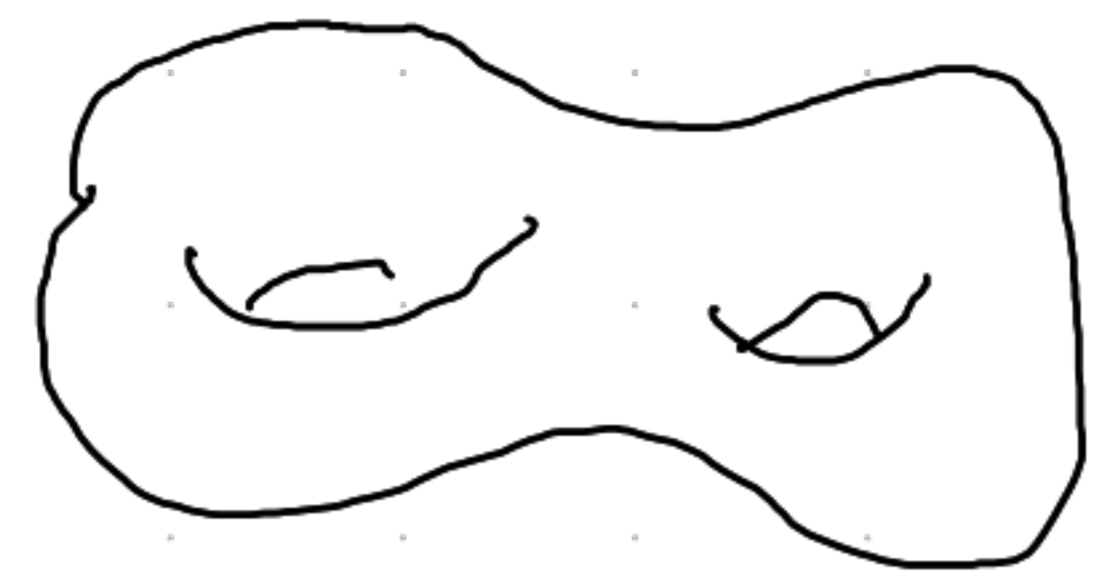
S^2

$g = 0$



T^2

$g = 1$



$g = 2$

$\mathbb{RP}^2$

Kleinuschel

$\mathbb{RP}^2 \# \mathbb{RP}^2$

They admit unique C^∞-structure but many holomorphic structures.

$\rightsquigarrow$ Theory of Riemann surfaces.

2.3 Partitions of unity

To extend local constructions / locally defined objects to global ones we need a method to glue them.

Def. 2.26 M manifold.

① Γ