Def. 2.26 M manifold.

(1) For a smooth fct. $f: M \to \mathbb{R}^m$ the support of f is defined as

$$\operatorname{supp}(f) := \overline{\{x \in M : f(x) \neq 0\}}$$

(2) A (smooth) partition of unity on M is a family $\mathcal{F} = \{f_i : M \to \mathbb{R} : i \in I\}$ of smooth real-valued fcts satisfying:

(a) $\mathcal{F}$ is locally finite: for all $x \in M$ $\exists$ an open neighbhd. U_x ~~of~~ $\subseteq M$ of x s.t. $\{i \in I : \operatorname{supp}(f_i) \cap U_x \neq \emptyset\}$ is finite.

(b) Any $f \in \mathcal{F}$ has values in $[0,1]$.

(c) For any $x \in M$, $\sum_{i \in I} f_i(x) = 1$ (sum is finite by (a)).

(2) An <u>open cover of M</u> is a family $\mathcal{U} = \{U_j : j \in J\}$ of open subsets $U_j \subseteq M$ s.t. $M = \bigcup_{j \in J} U_j$.

A partition of unity $\mathcal{F} = \{f_i : M \to \mathbb{R} : i \in I\}$ on M is subordinate to $\mathcal{U}$, if for every $i \in I$ $\exists j \in J$ s.t. $\operatorname{supp}(f_i) \subset U_j$.

<u>Thm 2.27</u> Suppose M is a smooth manifold and $\mathcal{U} = \{U_i : i \in I\}$ an open cover of M. Then $\exists$ a (smooth) partition of unity of countably many fcts $\mathcal{F} := \{f_k : M \to \mathbb{R} : k \in \mathbb{N}\}$ subordinate.

Remark.

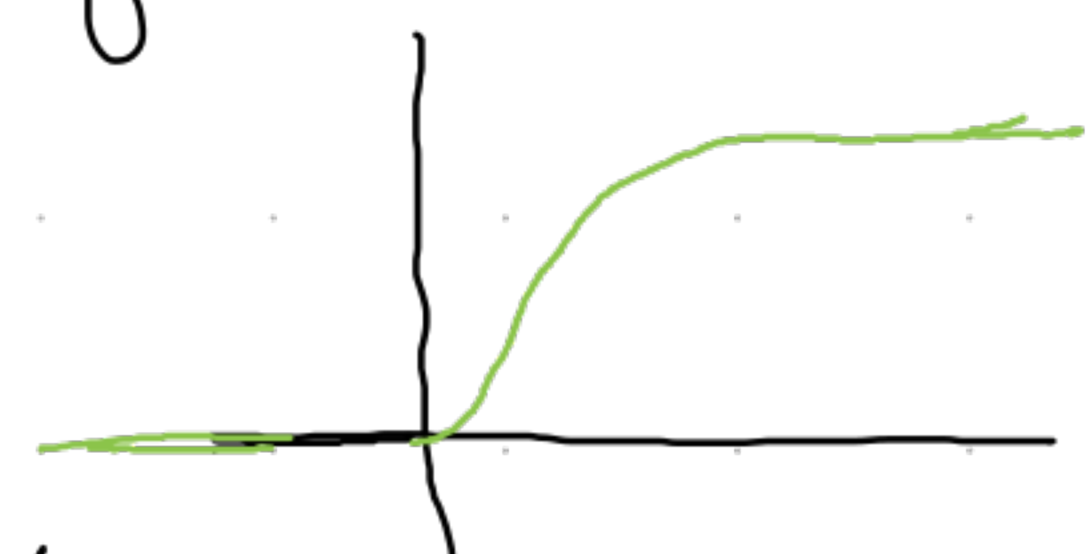
- Topolog. assumptions (M Hausdorff + 2nd countable) are essential for Thm. 2.27 to hold.

Apart from topolog. assumptions, key to Thm. 2.27 is:

Lemma 2.28 For any $x_0 \in \mathbb{R}^n$ and any open neighbhd. $U \subseteq \mathbb{R}^n$ of x_0 $\exists$ C^∞-fct. $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{supp}(f) \subseteq U$, $f \geq 0$ and $f(x_0) > 0$.

Proof Let $\rho : \mathbb{R} \to \mathbb{R}$ be the following fct.

$$\rho(t) := \begin{cases} e^{-\frac{1}{t^2}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$



It is smooth (see calculus class) (but not real-analytic).

$\exists \varepsilon > 0$ s.t. $B_{2\varepsilon}(x_0) = \{ x \in \mathbb{R}^n : \|x - x_0\| < 2\varepsilon \} \subset U$.

$f : \mathbb{R}^n \to \mathbb{R}$

$$f(x) = \rho\left(\varepsilon^2 - \langle x - x_0, x - x_0 \rangle\right) = \rho\left(\underline{\varepsilon^2 - \|x - x_0\|^2}\right)$$

- $f(x) \geq 0 \ \forall x \in \mathbb{R}^n$, since $\rho \geq 0$.
- $f(x) > 0 \iff x \in B_\varepsilon(x_0)$; in particular, $f(x_0) > 0$.
- $\mathrm{supp}(f) = \{ x \in \mathbb{R}^n : \|x - x_0\| \leq \varepsilon \} \subset U$.

<u>Remark</u> $\nexists$ partitions of unity of holom. fcts on complex manifolds.