Exam dates :

- Friday, 5. Febraury 2021
- Thursday, 11. Februrary 2021

Exam will have two parts :

- Written part, 2 hours (morning)
- oral exam, ~ 1/2 hours

- End of the class : 12.1. 2021

$(M, g) \subseteq (\mathbb{R}^{n+1}, g_{euc} = \langle , \rangle)$ hypersurface.

<u>Levi-Civita connection and Riemann curvature</u>:

$\eta \in \Gamma(TM)$ $\underset{\text{may be viewed as smooth fct}}{\leadsto}$ $\eta : M \longrightarrow \mathbb{R}^{n+1}$, $(\eta(x) \in T_xM \subseteq T_x\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$

and as such we can form its derivative in direction of another vector field $\xi \in \Gamma(TM)$:

$$\xi \cdot \eta(x) = T_x\eta\, \xi_x$$

In general, $\xi \cdot \eta(x) \in T_x\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$ is not an element in T_xM.

But we can project $(\xi\cdot\eta)(x)$ orthogonally w.r.to $g_{euc} = \langle , \rangle$ to T_xM $\quad$ ($T_x\mathbb{R}^{n+1} = T_xM \oplus (T_xM)^\perp$.)

We write $(\nabla_\xi \eta)(x) \in T_xM$ for the resulting element.

If ν is a local unit normal v.f. defined around $x \in M$, then

$$\boxed{(\nabla_\xi \eta)(x) = (\xi\cdot\eta)(x) - \langle (\xi\cdot\eta)(x), \nu(x)\rangle \nu(x)} .$$

The formula shows that $\nabla_\xi \eta$ is a vector field on M.

Def. 6.12 $(M,g) \subseteq (\mathbb{R}^{n+1}, \langle\, ,\rangle)$

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$
$$(\xi, \eta) \longmapsto \nabla_\xi \eta$$

is called the Levi-Civita connection of (M,g).

(or covariant deriv. of (M,g)).

For $\eta \in \Gamma(TM)$, $0 = \langle \eta, \nu \rangle$ and differentiating in direction of $\xi \in \Gamma(TM)$ gives: $0 = \langle \xi \cdot \eta, \nu \rangle + \underbrace{\langle \eta, \xi \cdot \nu \rangle}_{\mathrm{II}(\eta,\xi) = \mathrm{II}(\xi,\eta)}$

$$\Longrightarrow \quad \boxed{\nabla_\xi \eta = \xi \cdot \eta + \mathrm{II}(\xi,\eta)\nu} \qquad \text{(Gauß equation).}$$

Prop. 6.13 $(M,g) \subseteq (\mathbb{R}^{n+1}, \langle , \rangle)$ hypersurface, $\xi, \eta, e \in \Gamma(TM)$ and $f \in C^\infty(M,\mathbb{R})$.

Then $\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ has the following properties:

1. ∇ is bilinear over $\mathbb{R}$.

2. $\nabla_{f\xi}\eta = f\nabla_\xi \eta$ and $\nabla_\xi(f\eta) = (\xi \cdot f)\eta + f\nabla_\xi \eta$.

   (i.e. $\nabla\eta \in \Gamma(T^*M \otimes TM)$)

3. $\nabla_\xi \eta - \nabla_\eta \xi = [\xi,\eta]$

④ $\xi \cdot g(\eta, \epsilon) = g(\nabla_\xi \eta, \epsilon) + g(\eta, \nabla_\xi \epsilon)$ (∇ is compatible with g).

<u>Proof</u>

① $\xi \cdot \eta$ is bilinear over $\mathbb{R}$ and so is $\mathrm{II}(-,-)$, which implies the claim by Gauss eq.

② $(f\xi)\cdot\eta = f(\xi\cdot\eta)$ and $\xi\cdot(f\eta) = (\xi\cdot f)\eta + f(\xi\cdot\eta)$

Since II is a $\binom{0}{2}$-tensor (i.e. linear over $C^\infty(M,\mathbb{R})$), result follows from Gauss eq.

③ Proof of Prop. 6.7 shows $\xi\cdot\eta - \eta\cdot\xi = [\xi,\eta]$ and symmetry of $\mathrm{II}(-,-)$.

④ $\xi \cdot g(\eta, e) = \langle \underline{\xi \cdot \eta}, e \rangle + \langle \eta, \xi \cdot e \rangle$

$$= \langle \nabla_\xi \eta, e \rangle + \langle \eta, \nabla_\xi e \rangle$$

<u>Rework</u>: $\swarrow$ undef.

$c : I \longrightarrow M$

$c'(t) \in T_{c(t)}M$

$$c''(t) = \lim_{h \to 0} \frac{\underline{c'(t+h)} - c'(t)}{h}$$

Thm. 6.14 Suppose (M, g^M) and (N, g^N) are hypersurf. of $(\mathbb{R}^{n+1}, \langle , \rangle)$. with Levi-Civita connections ∇^M and ∇^N. If $f : (M, g^M) \to (N, g^N)$ is an isometry, then

$$f^*(\nabla^N_\xi \eta) = \nabla^M_{f^*\xi} f^*\eta \qquad \forall \xi, \eta \in \Gamma(TN).$$

Proof $A(\xi, \eta) := f^*(\nabla^N_\xi \eta) - \nabla^M_{f^*\xi} f^*\eta$.

- A is symmetric:

$$f^*(\nabla^N_\xi \eta) - f^*(\nabla^N_\eta \xi) = f^*(\nabla^N_\xi \eta - \nabla^N_\eta \xi) \underset{\substack{\text{Prop. 6.13} \\ 3}}{=} f^*([\xi, \eta])$$

$$\underset{\substack{\text{Prop. 6.13} \\ 3}}{=} [f^*\xi, f^*\eta] = \nabla^M_{f^*\xi} f^*\eta - \nabla^M_{f^*\eta} f^*\xi$$

$\Longrightarrow \quad A(\xi, \eta) - A(\eta, \xi) = 0$

- $\varphi \in \Gamma(TN)$, then 4 of Prop. 6.13 :

$$(*) \quad \underline{f^*\xi \cdot g^M(f^*\eta, f^*\varphi)} = g^M(\nabla^M_{f^*\xi} f^*\eta, f^*\varphi) + g^M(f^*\eta, \nabla^M_{f^*\xi} f^*\varphi)$$

Since f is an isometry, $g^M_x((f^*\eta)(x), (f^*\varphi)(x))$

$$= g^N_{f(x)}(\eta(f(x)), \varphi(f(x))).$$

i.e, $\underline{g^M(f^*\eta, f^*\varphi) = g^N(\eta, \varphi) \circ f}$.

LHS of (*) $= (f^* \xi)\cdot\left(g^N(\eta, e)\circ f\right) = \left(\xi\cdot g^N(\eta, e)\right)\circ f$

$$\underset{\text{(T4) of Prop. 6.13}}{=} g^N(\nabla^N_\xi \eta, e)\circ f + g^N(\eta, \nabla^N_\xi e)\circ f$$

$$= g^M\left(f^*(\nabla^N_\xi \eta), f^* e\right) + g^M\left(f^*\eta, f^*(\nabla^N_\xi e)\right)$$

Subtracting this identity from (*) gives:

$$0 = g^M\left(A(\xi,\eta), f^* e\right) + g^M\left(f^*\eta, A(\xi, e)\right) \quad \forall \xi, \eta, e \in \Gamma(TN)$$

$$\Rightarrow \underline{g^M\left(A(\xi,\eta), f^* e\right)} = g^M\left(A(\eta,\xi), f^* e\right) = -g^M\left(f^*\xi, A(\eta, e)\right)$$

$$= -g^M\left(f^*\xi, A(e,\eta)\right) = g^M\left(A(e,\xi), f^*\eta\right) = g^M\left(A(\xi,e), f^*\eta\right)$$

$= - g^M (f^* e, A(\xi, \eta)) \implies g^M (A(\xi, \eta), f^* e) = 0$

Since any tangent vector at x of M can be written as $f^* e(x)$ for an appropr. $e \in \Gamma(TN)$, this implies $A(\xi, \eta) = 0$ by non-degeneracy of g^M.

$\square$.

Hence, we see the Levi-Civita connection of (M, g) is intrinsic and so are all quantities and objects which can be written in terms of g and ∇.

For $\xi, \eta, \zeta \in \Gamma(TM)$ we have :

$$(**) \qquad \underline{\xi\cdot(\eta\cdot\zeta)} - \underline{\eta\cdot(\xi\cdot\zeta)} - \underline{[\xi,\eta]\cdot\zeta} = 0$$

LHS has a part tangential to M and a part orthog. to M and both need to vanish in order for $(**)$ to hold.

- $\eta\cdot\zeta = \nabla_\eta \zeta - \mathrm{II}(\eta,\zeta)\nu = \nabla_\eta\zeta - \underline{g(L(\eta),\zeta)}\nu$.

$$\underline{\xi\cdot(\eta\cdot\zeta)} = \underline{\xi\cdot\nabla_\eta\zeta} - \underbrace{(\xi\cdot g(L(\eta),\zeta))}\nu - \underline{g(L(\eta),\zeta)L(\xi)}$$

$$- g(\nabla_\xi L(\eta),\zeta)\nu$$
$$- g(L(\eta),\nabla_\xi\zeta)\nu.$$

- $[\xi,\eta]\cdot e = \nabla_{[\xi,\eta]} e - g(L([\xi,\eta]), e)\,\nu.$

$\Longrightarrow$ $(**)$ is equivalent to

- $$\underbrace{\nabla_\xi \nabla_\eta e - \nabla_\eta \nabla_\xi e - \nabla_{[\xi,\eta]} e}_{=:\, R(\xi,\eta)(e).} = \underline{\mathrm{II}(\eta,e)\,L(\xi) - \mathrm{II}(\xi,e)\,L(\eta)}$$

- $$-g(\nabla_\xi L(\eta), \underline{e})\,\nu + g(\nabla_\eta L(\xi), e)\,\nu + g(L[\xi,\eta], e)\,\nu = 0$$

$\Longleftrightarrow$ $L([\xi,\eta]) = \nabla_\xi L(\eta) - \nabla_\eta L(\xi)$

Mainardi Equation

Def 6.15 $(M,g) \subset (\mathbb{R}^{n+1}, \langle , \rangle)$ hypersurface, ∇ Levi-Civita connection of (M,g).

The Riemann curvature (tensor) of (M,g) is the $\binom{1}{3}$-tensor given by:

$$R : \Gamma(TM) \times \Gamma(TM) \longrightarrow L(\Gamma(TM), \Gamma(TM))$$
$$(\xi,\eta) \longmapsto (\mathrm{e} \mapsto R(\xi,\eta)(\mathrm{e}))\,.$$

From the formula $R(\xi,\eta) = -R(\eta,\xi)$.

Hence, $R \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(TM)) = \Gamma(\Lambda^2 T^*M \otimes T^*M \otimes TM)$

Thm 6.16 (Theorema egregium)

Suppose $(M, g) \subset (\mathbb{R}^3, \langle , \rangle)$ is a surface, $x \in M$, $\{\xi, \eta\} \in T_xM$ orthonormal basis of T_xM (w.r.t. to g).

Then

$$K(x) = g_x(R_x(\xi, \eta)(\eta), \xi).$$

In particular, the Gauss curvature is intrinsic.

If $\sigma = a\xi + b\eta$, $\tau = c\xi + d\eta$ are arbitrary vectors of T_xM then $R(\sigma, \tau)$ in the basis $\{\xi, \eta\}$ is given by

$$(ad - bc)\begin{pmatrix} 0 & K(x) \\ -K(x) & 0 \end{pmatrix}.$$

Proof.

$$R(\xi,\eta)(\eta) = \mathrm{II}(\eta,\eta)L(\xi) - \mathrm{II}(\xi,\eta)L(\eta)$$
$$= g(L(\eta),\eta)L(\xi) - g(L(\xi),\eta)L(\eta)$$

$$\Rightarrow g_x(R(\xi,\eta)(\eta),\xi) = g_x(L(\xi),\xi)\,g_x(L(\eta),\eta) - g_x(L(\xi),\eta)\,g_x(L(\eta),\xi).$$

RHS = determinant of L in the basis $\{\xi,\eta\}$ $= K(x)$.

From $g_x(R_x(\xi,\eta)\eta,\eta) = 0$ follow $R_x(\xi,\eta)(\eta) = K(x)\xi$.

and $R(\xi,\eta)(\xi) = -R(\eta,\xi)\xi = -K(x)\eta$.

By skew-symmetry and bilinearity over $\mathbb{R}$ this implies the formula for general σ and τ.

$\square$.

Significance of the Riemann curvature of $(M,g) \subseteq (\mathbb{R}^{n+1}, \langle , \rangle)$.

(M,g) is locally isometric around any point $x \in M$ to an open subset of $(\mathbb{R}^n, g) \iff R \equiv 0$ $(R_x = 0 \ \forall x \in M)$

In particular, a surface in $\mathbb{R}^3$ is locally isometric to an open subset of $\mathbb{R}^2 \iff K(x) = 0 \ \forall x \in M$.