Geodesics in hypersurfaces

$(M, g) \subseteq (\mathbb{R}^{n+1}, \langle , \rangle)$ hypersurface.

Def. 6.17 A (smooth) curve $c: I \to M$, $I \subseteq \mathbb{R}$ interval, is called a geodesic of (M, g), if $\forall t \in I$ the acceleration $c''(t)$ (taken as a curve in $\mathbb{R}^{n+1}$) is orthogonal to $T_{c(t)}M \subseteq T_{c(t)}\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$.

- If $(M = \mathbb{R}^n, g) \subseteq (\mathbb{R}^{n+1}, \langle , \rangle)$, the geodesics are the affine lines in $\mathbb{R}^n$: $c(t) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \\ 0 \end{pmatrix} + t \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ 0 \end{pmatrix} \in \mathbb{R}^n$

  ($c(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \\ 0 \end{pmatrix}$ and $c''(t) \perp \mathbb{R}^n \iff c_i''(t) = 0 \quad \forall t$.

In various ways geodesics in (M, g) are analogues of affine lines in $(\mathbb{R}^4, g)$:

- $c''(t) \perp T_{c(t)}M$ means that all occurring accelerations only amount to keep the curve c in M. These are the paths particles in M take when they are in free fall (no force is acting on it).

- $c''(t) \perp T_{c(t)}M \iff \underbrace{c''(t) - \langle c''(t), \nu(c(t)) \rangle \nu(c(t))}_{} = 0$

  where ν is a local unit normal vector field.

Since c is a curve in M, $\langle c'(t), \nu(c(t))\rangle = 0$ and different. in t yields: $-\langle c''(t), \underline{\nu(c(t))}'\rangle$

$$= \langle c'(t), T_{c(t)}\nu\, c'(t)\rangle$$
$$= \langle c'(t), L_{c(t)} c'(t)\rangle$$
$$= \mathrm{II}(c'(t), c'(t)).$$

$c: I \to M$ is a geod. $\Longleftrightarrow$ c is a solution of the second order ODE

$$(*) \qquad c''(t) + \mathrm{II}\big(c'(t), c'(t)\big)\nu(c(t)) = 0.$$

By viewing ν as a fct. defined on an open subset of $\mathbb{R}^{k+1}$

(*) is 2nd order ODE on open subset of $\mathbb{R}^{n+1}$.

Theory of ODE's $\Longrightarrow$ that for $x \in M$, $\xi_x \in T_x M$

$\exists$ locally a unique solution $c : I \longrightarrow \mathbb{R}^{n+1}$ of (*) with

$c(0) = x$, $c'(0) = \xi_x$ (I interval containing 0).

It is not hard to see that $c(t) \in M$ $\forall t$.

Hence, for any $x \in M$, $\xi_x \in T_x M$ $\exists$ a unique maximal geodesic $c : I \to M$ s.t. $c(0) = x$ and $c'(0) = \xi_x$.

• Relation to ∇ : $\eta \in \Gamma(TM)$, $c : I \to M$ C^∞-curve.

Then $\nabla_{\underline{c'(t)}} \eta (c(t))$ makes sense, (since $\nabla_s \eta(x)$ just depends on $s(x)$)

Moreover, $c'(t) \cdot \eta = \frac{d}{dt} \eta(c(t))$

It follows that $(\nabla_{c'} \eta)(c(t))$ just depends on $\eta\big|_{\mathrm{Im}(c)}$.

If η is a vector field along c (i.e. $\eta : I \to TM$ C^∞-map $\eta(t) \in T_{c(t)}M$).

then $\nabla_{c'} \eta$ is defined and again a smooth vector field along c. $((\nabla_{c'} \eta)(t) = (\nabla_{c'(t)} \eta)(c(t)))$

In particular, we may form $\nabla_{c'} c'$.

Since $c' \cdot c' = c''$ (by construction), the Gauss eq. implies LHS of $(*)$ $= \nabla_{c'} c'$

Geodesic equation $(*)$ can be written as $\nabla_{c'} c' = 0$

In particular, geodesics are intrinsic.

Ex. $S^n \subseteq \mathbb{R}^{n+1}$ $\quad (S^n, g_{rd}) \subseteq (\mathbb{R}^{n+1}, \langle \cdot , \cdot \rangle)$. $\quad c(t)$

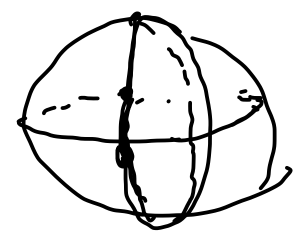
$$T_{c(t)} S^n = c(t)^{\perp}$$

Geodesic equation:

$$c''(t) + \mathrm{II}\big(c'(t), c'(t)\big)\, c(t) = c''(t) - \langle c''(t), c(t) \rangle c(t) = 0$$

c is a geod. $\iff c''(t) = \langle c''(t), c(t) \rangle\, c(t)$.

Geodesics are great circles.



Let $(x, \xi_x) \in TS^n$

$$c : t \longmapsto \begin{cases} x & \text{if } \xi_x = 0 \\ \cos(\|\xi_x\| t)\, x + \sin(\|\xi_x\| t) \dfrac{\xi_x}{\|\xi_x\|} & \text{if } \xi_x \neq 0. \end{cases}$$

$$(r, s) \longmapsto r x + s \frac{\xi_x}{\|\xi_x\|}$$

$$c(t) = \cos(\|\xi_x\| t)\, x + \sin(\|\xi_x\| t) \frac{\xi_x}{\|\xi_x\|}$$

$$c'(t) = -\|\xi_x\| \sin(\|\xi_x\| t)\, x + \|\xi_x\| \cos(\|\xi_x\| t) \frac{\xi_x}{\|\xi_x\|}$$

$$c''(t) = -\|\xi_x\|^2 \cos(\|\xi_x\| t)\, x - \|\xi_x\|^2 \sin(\|\xi_x\| t) \frac{\xi_x}{\|\xi_x\|}$$

$$= \underline{-\|\xi_x\|^2}\, c(t)$$

$$\langle c''(t), c(t) \rangle = -\|\xi_x\|^2 \underbrace{\left(\underline{\cos^2(\|\xi_x\| t)} \overbrace{\langle x, x \rangle}^{1} + \sin^2(\|\xi_x\| t) \underbrace{\left\langle \frac{\xi_x}{\|\xi_x\|}, \frac{\xi_x}{\|\xi_x\|} \right\rangle}_{1} \right)}_{=1}$$

$$= -\|\xi_x\|^2$$

$\Longrightarrow$ Great circles are geod. and all geod. are great circles, by uniqueness.

6.3 Riemannian mfds

(M, g) Riemannian mfd.

⇝ $\exists!$ torsion-free affine connection ∇ on M that is compatible with g (Levi-Civita conn. of (M,g)).

⇝ curvature of (M,g) = curv. of ∇.

⇝ ∇ determines a distinguished class of curves (geodesics of ∇ or (M,g)); $\underline{\nabla_{c'} c' = 0}$.

6.3.1 Affine connections

Suppose M is a n-dim. mfd.

<u>Def 6.18</u> An affine connection on M is a linear connection on the tangent bundle $TM \to M$, that is a $\mathbb{R}$-bilinear map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

s.t. (a) $\nabla_{f\xi}\eta = f\nabla_\xi \eta \quad \forall \xi, \eta \in \Gamma(TM)$

(b) $\nabla_\xi(f\eta) = (\xi \cdot f)\eta + f\nabla_\xi\eta. \quad \forall f \in C^\infty(M, \mathbb{R})$.

(cf. properties (1) and (2) of Prop. 6.13).

<u>Remark</u> For $\eta \in \Gamma(TM)$, (a) means that $\nabla \eta$ is a $\binom{1}{1}$-tensor.

Hence we can also say that an affine connection is an $\mathbb{R}$-linear map $\nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$ satisfying (b).

∇ $\quad \eta \longmapsto (\xi \mapsto \nabla_\xi \eta)$.

An affine connection on M is a device that allows to differentiate vector fields in direction of vector fields and hence allows to talk about the acceleration of a curve.

<u>Remark</u>: $\mathcal{L}_\xi \eta$ is not analogue of directional deriv., since not tensorial in ξ.

Ex. $M = \mathbb{R}^n$ $(x^1, \dots, x^n)$ $\cdot\ \xi = \sum \xi^i \frac{\partial}{\partial x^i}$, $\eta = \sum \eta^i \frac{\partial}{\partial x^i}$

v.f. on $\mathbb{R}^n$.

$$\nabla_\xi \eta\,(x) := \sum_{i=1}^{n} \xi \cdot \eta^i(x) \frac{\partial}{\partial x^i}(x) \qquad \left(= \xi \cdot \eta(x) \right)$$

Defines affine connection, called the standard (flat) connection on $\mathbb{R}^n$.

In particular, $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$.

∇ equals also the Levi-Civita connection of $(\mathbb{R}^n, g_{euc})$ $\subseteq (\mathbb{R}^{n+1}, \langle , \rangle)$.

Ex: Levi-Civita connections of hypersurfaces we defined in previous section (see Prop. 6.13).

Any manifold admits an affine connection (we saw that any mfd admits a Riem. metric g and will see g determines a distinguished affine connection), which implies it admits many (since adding any $\binom{1}{2}$-tensor to a connection is again an affine connection).

Properties of ∇ imply: $U \subseteq M$ open subset. Then

$\nabla_\xi \eta\big|_U$ just depends on $\xi\big|_U$ and $\eta\big|_U$.

In local coordinates (U, u): $\xi, \eta \in \Gamma(TM)$, $\xi\big|_U = \sum_i \xi^i \frac{\partial}{\partial u^i}$

$\eta = \sum_i \eta^i \frac{\partial}{\partial u^i}$.

$$\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = \sum_{i=1}^{k} \Gamma^k_{ij} \frac{\partial}{\partial u^k} \qquad \Gamma^k_{ij} : U \to \mathbb{R}$$

C^∞-fcts.

"connection coefficients of ∇ w.r.to (U, u)"

(or Christoffel symbols of —"—).

$$(\nabla_\xi \eta)\big|_U = \nabla_{\xi|_U} \eta|_U = \nabla_{\sum_i \xi^i \frac{\partial}{\partial u^i}} \sum \eta^j \frac{\partial}{\partial u^j} =$$

$$= \sum_{i,j} \xi^i \nabla_{\frac{\partial}{\partial u^i}} \left(\eta^j \frac{\partial}{\partial u^j}\right) = \sum_{i,j} \xi^i \frac{\partial \eta^j}{\partial u^i} \frac{\partial}{\partial u^j} + \sum_{i,j,k} \xi^i \eta^j \Gamma_{ij}^k \frac{\partial}{\partial u^k} .$$

Remark For a chart (U, u) a tensor Γ^U on U:

$\Gamma^U : \Gamma(TU) \times \Gamma(TU) \to \Gamma(TU)$ is given by

$$\Gamma^U(\xi, \eta) := \sum_{i,j,k} \xi^i \eta^j \Gamma_{ij}^k \frac{\partial}{\partial u^k} .$$

$$(\nabla_\xi \eta)\big|_U = \sum_j \xi \cdot \eta^j \frac{\partial}{\partial u^j} + \Gamma^U(\xi, \eta) .$$

$\boxed{\text{But } \Gamma_{ij}^k \text{ is not tensor on } M \text{ !}}$

Suppose $c: I \to M$ C^∞-curve and $\eta \in \mathcal{X}(M)$, then

the the local formula shows that

$$\left(\nabla_{c'(t)} \eta\right)(c(t)) \in T_{c(t)}M$$

just depends on the restriction of η to Image(c)

$$\left(c'(t) \cdot \eta^j (c(t))\right) = \frac{d}{dt} \eta^j (c(t)) .$$

Hence, if η is a vector field along c (i.e. $\eta : I \to TM$ C^∞-map s.t. $\eta(t) \in T_{c(t)}M$).

we may define $\left(\nabla_{c'} \eta\right)(t) := \left(\nabla_{c'(t)} \tilde{\eta}\right)(c(t))$

Where $\tilde{\eta}$ is a vector on a neigh. of $c(I) \subseteq M$ s.t. $\tilde{\eta}(c(t)) = \eta(t)$.

Then $\nabla_{c'}\eta$ is a well-def. vector field along c (indep. of the extension $\tilde{\eta}$).

Then the set of vector fields along $c: I \to M$ is a vector space and a module over C^∞-fct. $f: I \to \mathbb{R}$.

Lemma 6.19 (M, ∇) mfd. with affine connection, $c: I \to M$ is a C^∞-curve.

① Then the induced map $\nabla_{c'}$ from v.f. along c to v.f. along c is $\mathbb{R}$-linear and $\nabla_{c'} f\eta = f'\eta + f\nabla_{c'}\eta$.

(2) If $c : I \to M$ has values in a chart (U, u) and η is a vector field along c, then

$$\left(\nabla_{c'}\eta\right)(t) = \sum_{i=1}^{n} (\eta^i)'(t) \frac{\partial}{\partial u^i}(c(t)) + \Gamma^U\big(c'(t), \eta(c(t))\big)(c(t))$$

$$= \sum_{i=1}^{n} (\eta^i)'(t) \frac{\partial}{\partial u^i}(c(t)) + \sum_{i,j,k} (c^i)'(t)\, \eta^j(t)\, \Gamma_{ij}^k(c(t)) \frac{\partial}{\partial u^k}(c(t))$$

where $\eta(t) = \sum_{i=1}^{n} \eta^i(t) \frac{\partial}{\partial u^i}(c(t))$ $(\eta^i : I \to \mathbb{R}$ C^∞-fcts.)

and $c^i = u^i \circ c : I \to \mathbb{R}$ (C^∞-fct).

<u>Proof:</u> $\mathbb{R}$-linearity follows from $\mathbb{R}$-lin. of ∇. Local coordinate formula follows from local coord. formula of ∇ and also product rule follows easily from these.

<u>Def. 6.20</u> (M, ∇) mfd. with affine connection.

① A vector field $\eta \in \Gamma(TM)$ is called parallel w.r. to ∇, if $\nabla_\xi \eta = 0 \quad \forall \xi \in \Gamma(TM)$.

② If η is a vector field along a curve $c: I \to M$, then η is called parallel along c (w.r. to ∇), if $(\nabla_{c'} \eta)(t) = 0 \quad \forall t \in I$

1. defines an overdeterm. system of PDEs, hence in general there are no parallel vector fields.

2. of Lemma 6.19 shows that parallel v.f. along a fixed curve always exist, since they are solutions of a first order ODE.

Prop. 6.21 (M,∇) mfd. with affine connection

① Suppose $c : I \to M$ is C^∞-curve and $\eta_{c(t_0)} \in T_{c(t_0)}M$ a tangent vector at $c(t_0)$, where $t_0 \in I$.

Then $\exists!$ parallel vector field η along c s.t.

$$\eta(t_0) = \eta_{c(t_0)}.$$

② In the setting of ①, suppose $[t_0, t_1] \subseteq I$. Then

$$Pt_{t_0}^{t_1}(c) : T_{c(t_0)}M \to T_{c(t_1)}M$$
$$\eta_{c(t_0)} \mapsto \eta(t_1)$$

(where η is the vector field of ①).

is a linear isomorphism.

It is called the parallel transport along c determ. by ∇.

<u>Proof.</u>

① Suppose ① were already proved for curves with $c(I)$ is contained in the domain of a single chart.

By compactness for any $t_1 \in I$, $c([t_0, t_1])$ can be covered by finitely many charts in each of which η is defined by assumption and by uniqueness these defin. coincides on the intersection of the charts.

$\rightsquigarrow$ we get well-def. sol. η along $[t_0, t_1]$.

If $c(I)$ is contained in a chart (U, φ), then ② of Lemma 6.15 shows that $(\nabla_{c'}\eta)(t) = 0$ is equiv. to the system of first order linear ODEs:

$$\underline{(\eta^k)'(t)} + \sum_{i,j} (c^i)'(t)\, \underline{\eta^j(t)}\, \Gamma^k_{ij}(c(t)) = 0$$

$\forall k = 1, \dots, n.$

which implies ① and also ②.