Homework 2—Global Analysis

Due date:1.11.2020

1. Consider the general linear group $\mathrm{GL}(n,\mathbb{R})$ and the special linear group $\mathrm{SL}(n,\mathbb{R})$. We have seen that they are submanifolds of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ (even so called Lie groups) and that $T_{\mathrm{Id}}\mathrm{GL}(n,\mathbb{R}) \cong M_n(\mathbb{R}) = \mathbb{R}^{n^2}$.
   (a) Compute the tangent space $T_{\mathrm{Id}}\mathrm{SL}(n,\mathbb{R})$ of $\mathrm{SL}(n,\mathbb{R})$ at the identity Id.
   (b) Fix $A \in \mathrm{SL}(n,\mathbb{R})$ and consider the conjugation $\mathrm{conj}_A : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ by A given by $\mathrm{conj}_A(B) = A^{-1}BA$. Show that conj_A is smooth and compute the derivative $T_{\mathrm{Id}}\mathrm{conj}_A : T_{\mathrm{Id}}\mathrm{SL}(n,\mathbb{R}) \to T_{\mathrm{Id}}\mathrm{SL}(n,\mathbb{R})$.
   (c) Consider the map $\mathrm{Ad} : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{Hom}(T_{\mathrm{Id}}\mathrm{SL}(n,\mathbb{R}), T_{\mathrm{Id}}\mathrm{SL}(n,\mathbb{R}))$ given by $\mathrm{Ad}(A) := T_{\mathrm{Id}}\mathrm{conj}_A$. Show that Ad is smooth and compute $T_{\mathrm{Id}}\mathrm{Ad}$.

2. Consider $\mathbb{R}^n$ equipped with the standard inner product of signature (p,q) (where $p+q=n$) given by
$$\langle x, y\rangle := \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_i y_i$$
and the group of linear orthogonal transformation of $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ given by
$$\mathrm{O}(p,q) := \{A \in \mathrm{GL}(n,\mathbb{R}) : \langle Ax, Ay\rangle = \langle x, y\rangle \quad \forall x, y \in \mathbb{R}^n\}.$$
   (a) Show that
$$\mathrm{O}(p,q) = \{A \in \mathrm{GL}(n,\mathbb{R}) : A^{-1} = I_{p,q}A^t I_{p,q}\},$$
where $I_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$, and that $\mathrm{O}(p,q)$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension?
   (b) Show that $\mathrm{O}(p,q)$ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$ with respect to matrix multiplication μ and that $\mu : \mathrm{O}(p,q) \times \mathrm{O}(p,q) \to \mathrm{O}(p,q)$ is smooth (i.e. that $\mathrm{O}(p,q)$ is a Lie group.)
   (c) Compute the tangent space $T_{\mathrm{Id}}\mathrm{O}(p,q)$ of $\mathrm{O}(p,q)$ at the identity Id.