Homework 4—Global Analysis

Due date:1.12.2020

1. Let M be a (smooth) manifold and $\xi, \eta \in \mathfrak{X}(M)$ two vector fields on M. Show that

   (a) $[\xi, \eta] = 0 \iff (\mathrm{Fl}^{\xi}_t)^*\eta = \eta$, whenever defined $\iff \mathrm{Fl}^{\xi}_t \circ \mathrm{Fl}^{\eta}_s = \mathrm{Fl}^{\eta}_s \circ \mathrm{Fl}^{\xi}_t$, whenever defined.

   (b) If N is another manifold, $f : M \to N$ a smooth map, and ξ and η are f-related to vector fields $\tilde{\xi}$ resp. $\tilde{\eta}$ on N, then $[\xi, \eta]$ is f-related to $[\tilde{\xi}, \tilde{\eta}]$.

2. Consider the general linear group $\mathrm{GL}(n, \mathbb{R})$. For $A \in \mathrm{GL}(n, \mathbb{R})$ denote by

$$\lambda_A : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \qquad \lambda_A(B) = AB$$

$$\rho_A : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \qquad \rho_A(B) = BA$$

   left respectively right multiplication by A, and by $\mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ the multiplication map.

   (a) Show that λ_A and ρ_A are diffeomorphisms for any $A \in \mathrm{GL}(n, \mathbb{R})$ and that

$$T_B\lambda_A(B, X) = (AB, AX) \qquad T_B\rho_A(B, X) = (BA, XA),$$

   where $(B, X) \in T_B\mathrm{GL}(n, \mathbb{R}) = \{(B, X) : X \in M_n(\mathbb{R})\}$.

   (b) Show that

$$T_{(A,B)}\mu((A, B), (X, Y)) = T_B\lambda_A Y + T_A\rho^B X = (AB, AY + XB)$$

   where $(A, B) \in \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ and $(X, Y) \in M_n(\mathbb{R}) \times M_n(\mathbb{R})$.

   (c) For any $X \in M_n(\mathbb{R}) \cong T_{Id}\mathrm{GL}(n, \mathbb{R})$ consider the maps

$$L_X : \mathrm{GL}(n, \mathbb{R}) \to T\mathrm{GL}(n, \mathbb{R}) \qquad L_X(B) = T_{Id}\lambda_B(Id, X) = (B, BX).$$

$$R_X : \mathrm{GL}(n, \mathbb{R}) \to T\mathrm{GL}(n, \mathbb{R}) \qquad R_X(B) = T_{Id}\rho_B(Id, X) = (B, XB).$$

   Show that L_X and R_X are smooth vector field and that $\lambda_A^* L_X = L_X$ and $\rho_A^* R_X = R_X$ for any $A \in \mathrm{GL}(n, \mathbb{R})$. What are their flows? Are these vector fields complete?

   (d) Show that $[L_X, R_Y] = 0$ for any $X, Y \in M_n(\mathbb{R})$.

3. Suppose α^i_j for $i = 1, ..., k$ and $j = 1, ..., n$ are smooth real-valued functions defined on some open set $U \subset \mathbb{R}^{n+k}$ satisfying

$$\frac{\partial \alpha^i_j}{\partial x^k} + \alpha^\ell_k \frac{\partial \alpha^i_j}{\partial z^\ell} = \frac{\partial \alpha^i_k}{\partial x^j} + \alpha^\ell_j \frac{\partial \alpha^i_k}{\partial z^\ell},$$

where we write $(x, z) = (x^1, ..., x^n, z^1, ..., z^k)$ for a point in $\mathbb{R}^{n+k}$. Show that for any point $(x_0, z_0) \in U$ there exists an open neighbourhood V of x_0 in $\mathbb{R}^n$ and a unique C^∞-map $f : V \to \mathbb{R}^k$ such that

$$\frac{\partial f^i}{\partial x^j}(x^1, ..., x^n) = \alpha^i_j(x^1, ..., x^n, f^1(x), ..., f^k(x)) \quad \text{and} \quad f(x_0) = z_0.$$

In the class/tutorial we proved this for $k = 1$ and $j = 2$.

4. Which of the following systems of PDEs have solutions $f(x, y)$ (resp. $f(x, y)$ and $g(x, y)$) in an open neighbourhood of the origin for positive values of $f(0, 0)$ (resp. $f(0, 0)$ and $g(0, 0)$)?

   (a) $\frac{\partial f}{\partial x} = f \cos y$ and $\frac{\partial f}{\partial y} = -f \log f \tan y$.

   (b) $\frac{\partial f}{\partial x} = e^{xf}$ and $\frac{\partial f}{\partial y} = xe^{yf}$.

   (c) $\frac{\partial f}{\partial x} = f$ and $\frac{\partial f}{\partial y} = g$; $\frac{\partial g}{\partial x} = g$ and $\frac{\partial g}{\partial y} = f$.