Yesterday: M mfd., $E \subseteq TM$ smooth distribution.

When is E integrable?

A necessary condition is involutivity of E; i.e. E is closed under the Lie bracket.

We will show now that involutivity is also sufficient (Frobenius Thm.).

Note that involutivity is easy to check:

Lemma 3.36 Suppose $E \subseteq TM$ is a smooth distribution on a mfd M.
Then E is involutive $\iff$ locally around each point $x \in M$ $\exists$ a local frame $\{s_1, \dots, s_k\}$ s.t. $[s_i, s_j]$ is a local section of E $\forall i,j$

Proof Follows from ② of Prop. 3.29.

Recall that the coordinate vector fields $\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}$ corresp. to a chart (U, u) of M defines a local frame of TM and $[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] = 0$ $\forall i, j$. Note that $\{\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^k}\}$ $(k \leq n)$ span an integrable distribution on U $\leadsto$

integral submfds are given by $u^{-1}(y, a)$ for fixed $a \in u(U) \cap \mathbb{R}^{n-k}$ $\quad u(U) \subseteq \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.

<u>Lemma 3.37</u> Suppose M is a mfd. of dim. n, $V \subseteq M$ an open subset. If $\{\xi_1, \ldots, \xi_k\}$ are local vector fields on V s.t. $\xi_1(y), \ldots, \xi_k(y) \in T_yV = T_yM$ are linearly independent $\forall y \in V$, then the following are equivalent:

① $[\xi_i, \xi_j] = 0 \quad \forall i,j$

② For any $y \in V$ $\exists$ a chart (U,u) with $y \in U$ s.t.

$$\frac{\partial}{\partial u^1} = \xi_1, \ldots, \frac{\partial}{\partial u^k} = \xi_k .$$

<u>Proof.</u> ② $\Rightarrow$ ① ✓

① $\Longrightarrow$ ② :

Fix $y \in V$ and let $(\tilde{U}, \tilde{u})$ be a chart with $y \in \tilde{U}$, $\tilde{u}(y) = 0$

and $\xi_i(y) = \frac{\partial}{\partial \tilde{u}^i}(y)$ $i = 1, \dots, k$.

$\exists$ open neighbhds W and $\tilde{W}$ of 0 in $\mathbb{R}^k$ resp. $\mathbb{R}^{n-k}$ s.t.

$$\phi((t^1, \dots, t^k), (t^{k+1}, \dots, t^n)) = Fl^{\xi_1}_{t^1} \circ \dots \circ Fl^{\xi_k}_{t^k}(\tilde{u}^{-1}(0, (t^{k+1}, \dots, t^n)))$$

is defined $\forall (t^1, \dots, t^k) \in W$ and $(t^{k+1}, \dots, t^n) \in \tilde{W}$. $\left(\tilde{u} : \tilde{U} \to \underbrace{\tilde{u}(\tilde{U})}_{\mathbb{R}^k} \times \underbrace{\tilde{u}(0)}_{\mathbb{R}^{n-k}}\right)$

and a smooth map $\phi : W \times \tilde{W} \to M$.

By construction, $\phi(0, 0) = y$.

For $i \le k$ we have:

$$\frac{\partial \phi}{\partial t^i}(t) = \frac{d}{ds}\Big|_{s=0} \phi\left(t^1, \dots, t^i + s, \dots, t^n\right) = \quad (*)$$

$$= \frac{d}{ds}\Big|_{s=0} Fl_s^{\xi_i}\left(\phi(t)\right) = \xi_i\left(\phi(t)\right).$$

$Fl^{\xi_i}_{t^i+s} = Fl^{\xi_i}_{t^i} \circ Fl^{\xi_i}_s$ and $Fl^{\xi_i}_s$ commutes with all $Fl^{\xi_j}_{t^j}$

In particular, $\frac{\partial \phi}{\partial t^i}(0,0) = \xi_i(y)$
$= \frac{\partial}{\partial \tilde{u}^i}(y)$.

For $i > k$ and $t^1 = \dots = t^k = 0$ we have:

$$\frac{\partial \phi}{\partial t^i}(0) = \frac{d}{dt}\Big|_{t=0} \phi(te_i) = \frac{d}{dt^i}\Big|_{t^i=0} \phi(0, \dots, t^i, \dots, 0) =$$

$$= \frac{d}{dt^i}\Big|_{t^i=0} \tilde{u}^{-1}(0,\ldots 0, t^i, 0 \ldots 0) = T_{(a_0)}\tilde{u}^{-1} e^i = \frac{\partial}{\partial \tilde{u}^i}(y).$$

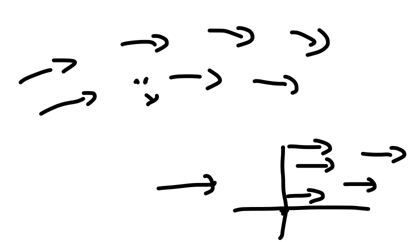
$\Rightarrow$ $T_{(0,0)}\phi$ is invertible $\left(\{\frac{\partial}{\partial t^i}(0)\}\right.$ is map to $\left.\{\frac{\partial}{\partial \tilde{u}^i}(y)\}\right)$

$\Rightarrow$ By possibly shrinking W and $\tilde{W}$ we may assume that $\phi : W \times \tilde{W} \to U$ is a diffeom., where U is an open neighbhd. of $y \in M$ and $u := \phi^{-1} : U \to W \times \tilde{W} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$ is the required chart. by $(*)$.

$\square$

Remark

If $\xi \in \mathfrak{X}(M)$, then for any $x \in M$ s.t. $\xi(x) \neq 0$, there exist a chart (U, u) with $x \in U$ s.t. $\xi|_U = \frac{\partial}{\partial u^1}$.



Thm. 3.38 (Frobenius Thm; local version).

Let M be a mfd of dim. n and $E \subseteq TM$ a smooth involutive distribution of rank k. Then for any $x \in M$ there exists a chart (U, u) with $x \in U$ s.t.

- $u(U) = W \times \tilde{W} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$ $\quad W \subseteq \mathbb{R}^k$, $\hat{W} \subseteq \mathbb{R}^{n-k}$ are open subsets.
- for each $a \in \tilde{W}$ the subset $u^{-1}(W \times \{a\}) \subseteq M$ is an integral mfd. for E.

In particular, any smooth involutive distribution is integrable.

Proof. We show that around each point there exists a local frame of commuting sections of E. Then Lemma 3.37 implies the Theorem.

Fix $x \in M$ and a local frame $\{s_1, \dots, s_k\}$ defined on an open neighbhd. $\tilde{U}$ of x. With loss of generality, we may assume $\tilde{U}$ is the domain of a chart $(\tilde{U}, \hat{u})$ with $\hat{u}(x) = 0$.

Then for $j = 1, \dots, k$ we have

$$s_j = \sum_{i=1}^{n} f_j^i \frac{\partial}{\partial \hat{u}^i} \qquad f_j^i \in C^\infty(\tilde{U}, \mathbb{R}).$$

Since $\{s_j(y)\}_{j=1}^{k}$ is a basis of E_y $\forall y \in \tilde{U}$, the $n \times k$ - matrix $\{f_j^i(y)\}_{j=1,\dots,k}^{i=1,\dots,n}$ has rank k $\forall y \in \tilde{U}$.

Renumbering the coordinates, we may assume that at x the first k-rows of $\{f^i_j(x)\}$ are linearly independent.

By continuity, this holds locally around x and so by possibly shrinking $\tilde{U}$ we may assume it holds on $\tilde{U}$.

For $y \in \tilde{U}$ let $\underline{(g^i_j(y))}$ be the inverse of $(f^i_j(y))^{i=1,\dots,k}_{j=1,\dots,k}$

Since inversion in $GL(k,\mathbb{R})$ is smooth, the fcts $g^i_j : \tilde{U} \to \mathbb{R}$ are smooth.

$$\Longrightarrow \quad \eta_i := \sum_{j=1}^{k} g^j_i s_j \qquad \text{for } i = 1, \dots, k$$

are (smooth) local sections of E defined on $\tilde{U}$.

Since $(g^i_j(y))$ is invertible $\forall y \in \tilde{U}$ and $\{s_1, \dots, s_k\}$ a local frame of E, also $\{\eta_1, \dots, \eta_k\}$ is a local frame of E on $\tilde{U}$.

<u>Claim</u>: $[\eta_i, \eta_j] = 0 \quad \forall i,j$.

(*) $\underline{\eta_i} = \sum_{j=1}^{k} g^j_i s_j = \sum_{\substack{1 \le e \le n \\ 1 \le j \le k}} g^j_i f^e_j \frac{\partial}{\partial \tilde{u}^e} = \frac{\partial}{\partial \tilde{u}^i} + \sum_{e > k} h^e_i \frac{\partial}{\partial \tilde{u}^e}$

$h^e_i \in C^\infty(\tilde{U}, \mathbb{R})$.

By involutivity, $\underline{[\eta_i, \eta_j]} = \sum_{m=1}^{k} c^m_{ij} \eta_m$. $\quad c^m_{ij} \in C^\infty(\tilde{U}, \mathbb{R})$.

$$\text{RHS} \underset{(*)}{=} \sum_{m=1}^{k} c^m_{ij} \left(\frac{\partial}{\partial \tilde{u}^m} + \sum_{e > k} h^e_m \frac{\partial}{\partial \tilde{u}^e} \right) = \sum_{m=1}^{k} c^m_{ij} \frac{\partial}{\partial \tilde{u}^m} + \sum_{m > e} \tilde{h}^e_{ij} \frac{\partial}{\partial \tilde{u}^e}$$

$\tilde{h}^e_{ij} \in C^\infty(\tilde{U}, \mathbb{R})$.

$$\text{LHS} \overset{(*)}{=} \Big[\underline{\frac{\partial}{\partial \tilde{u}^i}} + \sum_{e>k} h_i^e \underline{\frac{\partial}{\partial \tilde{u}^e}}, \frac{\partial}{\partial \tilde{u}^j} + \sum_{e>k} h_j^e \frac{\partial}{\partial \tilde{u}^e} \Big]$$

$$= \sum_{e>k} h_{ij}^e \frac{\partial}{\partial \tilde{u}^e} \qquad h_{ij}^e \in C^\infty(\tilde{U}, \mathbb{R})$$

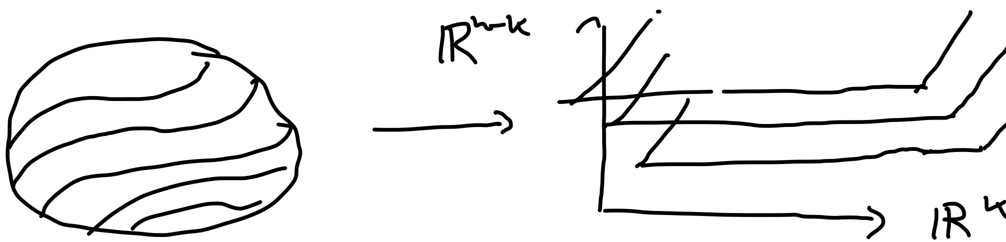
$$\Longrightarrow \left(\tilde{h}_{ij}^e = h_{ij}^e \right) \text{ and } \sum_{m=1}^{k} c_{ij}^m \frac{\partial}{\partial \tilde{u}^m} = 0 \text{ on } \tilde{U} \Longrightarrow c_{ij}^m = 0 \text{ on } \tilde{U}.$$

Hence, $[\eta_i, \eta_j] = 0$.

By Lemma 3.37 $\exists$ a chart $u : U \longrightarrow u(U) = W \times \tilde{W} \subseteq \mathbb{R}^k \times \underbrace{\mathbb{R}^{n-k}}_{= \mathbb{R}^l}$ with $x \in U$, $u(x) = (0,0)$ and we may also arrange that $U \subset \tilde{U}$ s.t. $\eta_i|_U = \dfrac{\partial}{\partial u^i}$ $i = 1, \dots, k$.

Hence, for each $a \in \tilde{W}$, $u^{-1}(W \times \{a\})$ is an integral manifold for E. (eq. for $u^{k+1} = a^{k+1}, \dots, u^n = a^n$). $\square$.

Thm 3.38 says that, given an involutive ~~...~~ smooth distribution, locally around any point there exist a chart (U, u) where U is filled by integral submanif.; in the corresp. coordinates they are given by the affine horizontal subspaces $\mathbb{R}^k \times \{a\}$ of $\mathbb{R}^n$.



Charts as in Thm. 3.38 are called distinguished charts for (M, E) (where E is smooth involutive distrib.) and the integral submfds. $u^{-1}(W \times \{a\}) \subseteq M$ are called plaques.

Note that, (U_α, u_α) and (U_β, u_β) are two such charts for (M, E) with $U_\alpha \cap U_\beta \neq \emptyset$, then the transition maps are of the form

$$\subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$$

$$\longrightarrow \quad u_\beta \circ u_\alpha^{-1} : \quad u_\alpha(U_\alpha \cap U_\beta) \longrightarrow u_\beta(U_\alpha \cap U_\beta) \qquad (*)$$

$$(x, y) \longmapsto (f(x,y), g(y))$$

for f, g smooth.

(i.e. transition maps maps plaques to plaques $\left(W_\alpha \times \{a\} \to W_\beta \times \{b\}\right)$).

<u>Def. 3.39</u> A foliated atlas of dim k on a mfd. $(M, \mathcal{A})$ of dim. n is a subatlas $\mathcal{A}'$ of $\mathcal{A}$ consisting of charts (U, u) s.t.

- $u(U) = W \times \tilde{W} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$ for open subsets $W \subseteq \mathbb{R}^k$ and $\tilde{W} \subseteq \mathbb{R}^{n-k}$
- transitions maps are of the form $(*)$

<u>Def. 3.40</u> A k-dimensional foliation $\mathcal{F}$ on a mfd M of dim. n is a maximal foliated atlas of dim. k.

Frobenius Thm. shows that any involutive smooth distribution E on M of dim. k defines a k-dim. foliation $\mathcal{F}^E$.

Conversely, any such foliation $\mathcal{F}$ determines a smooth involutive distribution of rank k on M:

$$E_x = T_{u(x)} u^{-1} \left(T_w \mathbb{R}^k \times \{0\} \right) \qquad u(x) = w + \tilde{w}$$

for a chart (U, u) of the foliation with $x \in U$. (by (*) E_x is independent of (U, u) with $x \in U$.)