Given def. of submfd. of $\mathbb{R}^n$, we have an obvious notion of smooth maps between them:

<u>Def 2.9</u> $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ submfds. of dim. k resp. ℓ.

(1) A map $f: M \to \mathbb{R}^m$ is smooth (C^∞), if for every $x \in M$ $\exists$ open neighbhd. U of x in $\mathbb{R}^n$ and a smooth map $\tilde{f}: \tilde{U} \to \mathbb{R}^m$ s.t. $\tilde{f}\big|_{\tilde{U} \cap M} = f\big|_{U \cap M}$.

(2) A map $f: M \to N$ is smooth, if it is smooth as a map $f: M \to \mathbb{R}^m$.

<u>Notation</u>: $C^\infty(M, N) := \{ f: M \to N,\ f \text{ smooth} \}$.

It follows: constant maps, the identity map and compositions of smooth maps are smooth.

<u>Def. 2.10</u> $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ submfds.

① A map $f: M \to N$ is a <u>diffeom.</u>, if f is a smooth bijection with smooth inverse.

We call M and N <u>diffeomorphic</u>, if $\exists$ a diffeom. between them. <u>Notation</u>: $M \simeq N$.

② A local diffeom. between M and N is a smooth map $f: M \to N$ s.t. for every $x \in M$ and $f(x) \in N$ $\exists$ open neighbhds U of x in M and V of $f(x)$ in N s.t. $f|_U : U \to V$ is a diffeom.

<u>Ex</u> $\mu : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ matrix multiplication

It is smooth $\mu(A,B) = AB$ $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$G = GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$ or $O(n) \subseteq M_n(\mathbb{R})$ submfd.

$\Longrightarrow$ $\mu : G \times G \longrightarrow G$ is smooth.

(G, μ) is a Lie group. $(\mu(A, i(A)) = \mathrm{Id}$

Implicit Fct Thm. implies $i : A \to A^{-1}$ is smooth$)$

Def. 2.11 $M \subseteq \mathbb{R}^n$ k-dim. submfd.

A (local) chart (or coordinate chart or local coordinates) for M is a diffeom. $u : U \longrightarrow V$, where $U \subseteq M$ open subset and $V \subseteq \mathbb{R}^k$ open subset.

Given a chart $u : U \to u(U) = V \subseteq \mathbb{R}^k$

$$u(x) = (u^1(x), \dots, u^k(x)) \in V \subseteq \mathbb{R}^k$$

Fcts $u^i : U \to \mathbb{R}$ are smooth and called the local coordinates associated with (u, U).

Lemma 2.12 $M \subseteq \mathbb{R}^n$ k-dim. submfd. Let $\psi : V \to \tilde{U}$ be a local parametrization, where $\tilde{U} \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^k$ are open. Then

$$u := \psi^{-1} : U \to V \qquad U := \tilde{U} \cap M$$

defines a chart of M. Conversely, given any chart $u : U \hookrightarrow V$, then $U = \tilde{U} \cap M$ with $\tilde{U} \subseteq \mathbb{R}^n$ open and $u^{-1} : V \to U \to \tilde{U}$ defines a local parametrization of M.

Proof. $\psi : V \to U = \tilde{U} \cap M$ is bijective, smooth as fct $V \to \tilde{U}$ and also a a fct. $V \to U$. Remains to show that $\psi^{-1} : U \to V$ is smooth.

By (4) $\Rightarrow$ (1) in Thm. 2.5: For any $x \in U = \tilde{U} \cap M$ $\exists U' \subseteq \mathbb{R}^k$ open neighbhd. of x in $\mathbb{R}^n$ and a $(n-k)$-dim. subspace $W^\perp \subseteq \mathbb{R}^n$ and an open neighbhd. V' of $(\psi^{-1}(x), 0)$ in $V \times W^\perp \subseteq \mathbb{R}^n$ s.t. $\Phi : V' \to U'$, $\Phi(y, w) := \psi(y) + w$

is a diffeomorphism. Hence, $\varphi^{-1} : U' \cap M \to \mathbb{R}^k$ is given by $pr_1 \circ \Phi^{-1}\big|_{U' \cap M}$, where $pr_1 : \underbrace{V}_{\subseteq \mathbb{R}^k} \times W^\perp \to V$ is the natural projection,

and so smooth as a composition of smooth maps. $\square$

Suppose $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ are submflds of dim k and l respectively. Let $f : M \to N$ be a continuous fct.
Fix $x \in M$ and let (U, u) be a chart of M with $x \in U$ and (V, v) a chart of N with $f(x) \in V$.
Set $W := u\big(f^{-1}(V) \cap U\big) \subseteq \mathbb{R}^k$, which is open.
Consider

$$v \circ f \circ u^{-1} : \underbrace{u(f^{-1}(V) \cap U)}_{\substack{\| \\ W \subseteq \mathbb{R}^k}} \to v(V) \subseteq \mathbb{R}^l$$

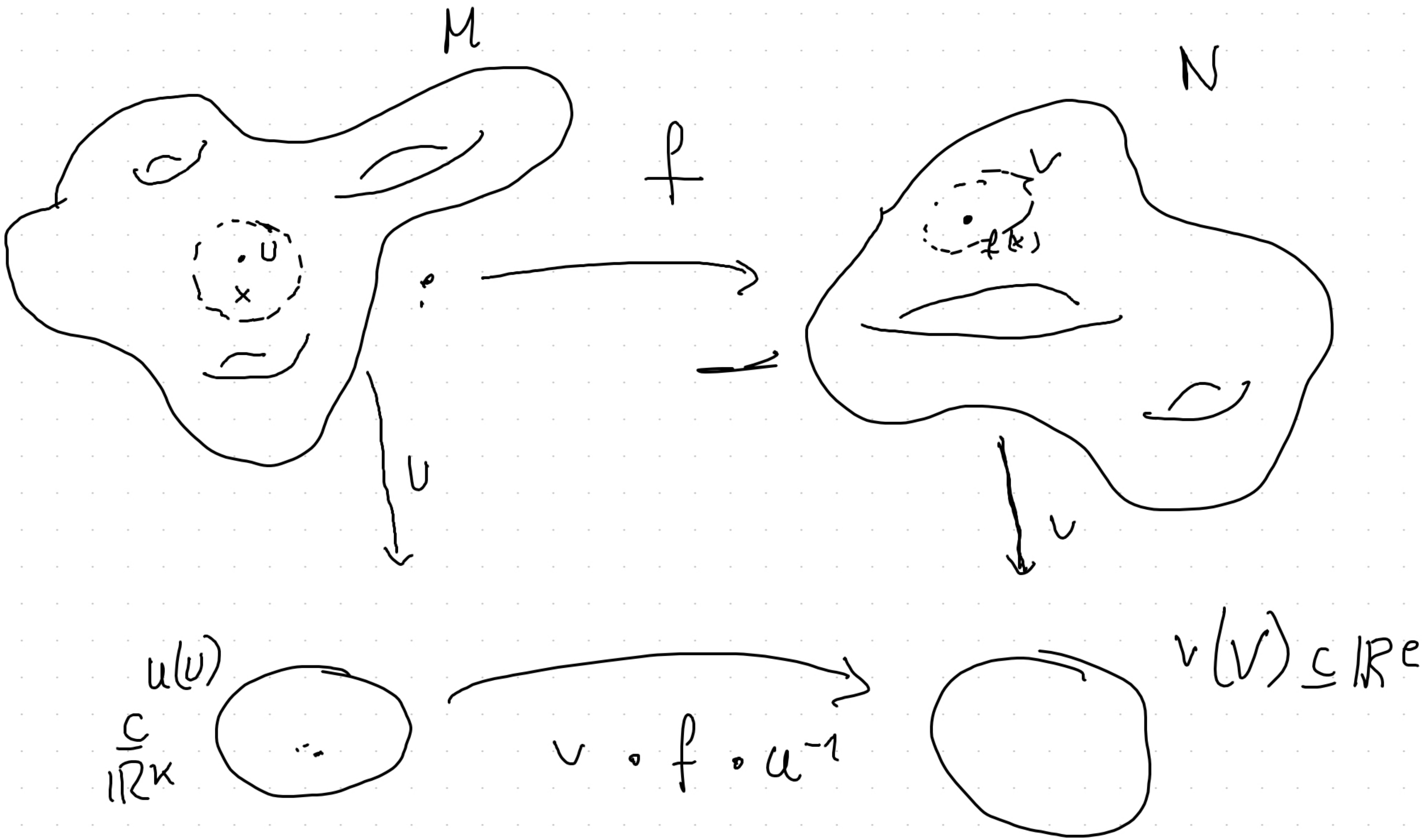
It is a continuous map between open subsets of $\mathbb{R}^k$ and $\mathbb{R}^e$.
With respect to (U,u) and (V,v) we can write

$$f = (f_1, \ldots, f_e),$$

where $v^j(f(y)) = f_j(u^1(y), \ldots, u^k(y))$ or short

$$v^j = f^j(u^1, \ldots, u^k).$$

<u>Def. 2.13</u> $(f_1, \ldots, f_e)$ is called the local coordinate expression of f with respect to (U,u) and (V,v).


M
N
f
U
x
u
V
f(x)
v
u(U) ⊆ ℝ^k
v ∘ f ∘ u^{-1}
v(V) ⊆ ℝ^e

Thm 2.14 $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ k-dim. resp. ℓ-dim. submfd.,
$f: M \longrightarrow N$ a map. Then the following are equivalent:

① f is smooth

② f is continuous and for every $x \in M$ $\exists$ charts (U,u) of M with $x \in U$ and (V,v) of N with $f(x) \in V$ s.t.

$$v \circ f \circ u^{-1} : u(f^{-1}(V) \cap U) \to v(V) \text{ is smooth.}$$

③ f is continuous and for every $x \in M$ and every chart (U,u) of M with $x \in U$ and every chart (V,v) of N with $f(x) \in V$ the composition $v \circ f \circ u^{-1}$ is smooth.

④ f is continuous and has in appropriate charts smooth local coordinate expressions.

⑤ f is continuous and has smooth local coordinate expressions with respect to any charts.

Proof

Evidently, 2 $\Longleftrightarrow$ 4 and 3 $\Longleftrightarrow$ 5 and also 3 $\Longrightarrow$ 2.

Since compositions of smooth maps are smooth, we have

1 $\Longrightarrow$ 3

Remains to show that 2 $\Longrightarrow$ 1.

Assume $v \circ f \circ u^{-1}$ is smooth. Since $u : U \to u(U) \subseteq \mathbb{R}^k$ is smooth, $\exists$ open subset $\tilde{U}$ of $\mathbb{R}^n$, $x \in \tilde{U}$ and $\tilde{u} : \tilde{U} \longrightarrow \mathbb{R}^k$ smooth s.t.

$\tilde{u}|_{U \cap \tilde{U}} = u|_{U \cap \tilde{U}}$.

Note that $\tilde{u}^{-1}(u(U)) \subseteq \mathbb{R}^n$ open and $\tilde{u} : \tilde{u}^{-1}(u(U)) \to u(U)$ smooth.

Set $\tilde{f} := v^{-1} \circ (v \circ f \circ u^{-1}) \circ \tilde{u} : \tilde{u}^{-1}(u(U)) \longrightarrow V$

$\Longrightarrow \tilde{f}$ is smooth as compos. of smooth maps and for $y \in \tilde{u}^{-1}(u(U)) \cap M$ we $\tilde{f}(y) = f(y)$

2.2 Abstract Manifolds

Def. 2.15 M is topological space.

① A chart with values in $\mathbb{R}^n$ on M is a homeomorphism $u: U \to u(U)$ from an open subset $U \subseteq M$ to an open subset $u(U) \subseteq \mathbb{R}^n$.

② A C^∞-atlas of charts with values in $\mathbb{R}^n$ on M is a set

$$\mathcal{A} = \{ (U_i, u_i) : i \in I \}$$

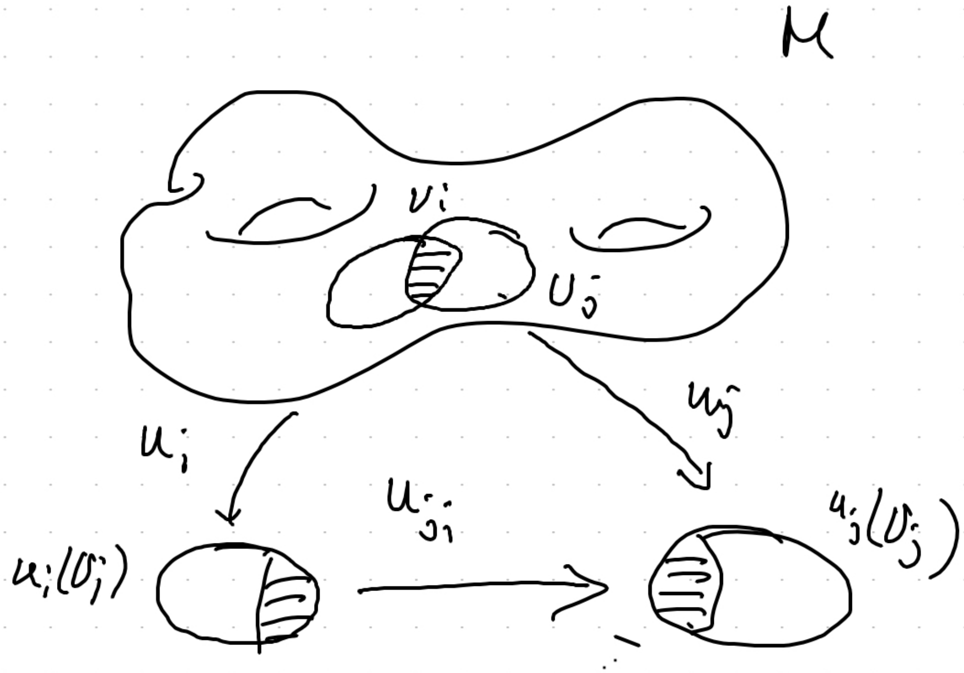
of charts with values in $\mathbb{R}^n$ s.t.

- $M = \bigcup_{i \in I} U_i$
- for any two charts (U_i, u_i) and (U_j, u_j) in A the map

$$u_{ji} := u_j \circ u_i^{-1} : \underbrace{u_i(U_i \cap U_j)}_{\subseteq \mathbb{R}^n \text{ open}} \longrightarrow \underbrace{u_j(U_i \cap U_j)}_{\subseteq \mathbb{R}^n \text{ open}}$$

is a diffeomorphism. (inverse u_{ij}).

transitions maps



Def. 2.16 Two atlases for a topological space M are equivalent (or compatible), if their union is again an atlas for M.

Lemma 2.17 Any atlas $\mathcal{A}$ on a topological space M is contained in a maximal one $\mathcal{A}_{max}$ given by the union of all ~~compatible~~ atlases compatible with $\mathcal{A}$.

Def. 2.18 A smooth (C^∞) manifold of dim. n is a second countable Hausdorff topological space equipped with a maximal C^∞-atlas with values in $\mathbb{R}^n$ (or an equivalence class of C^∞-atlases with values in $\mathbb{R}^n$).

Remark: • second countable $\Leftrightarrow$ M has a countable basis ($\exists$ countable collection of open sets $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ s.t.

any open subset of M is the union of subfamily of $\mathcal{U}$.