Recall: we defined the tangent at x, T_xM, of a submfd. $M \subseteq \mathbb{R}^n$.

Suppose $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ submfds. and $f: M \to N$ smooth map. The tangent map of f at x should be a linear map:

$$T_xf : T_xM \longrightarrow T_{f(x)}N.$$

~~In view of~~ ① of Prop. 3.1 and the fact that we want the chain rule to hold suggests the following definition:

$$T_xf\big(c(0), c'(0)\big) := \big(f\circ c(0), (f\circ c)'(0)\big) \in T_{f(x)}N. \quad (*)$$

where $c(0) = x$, $(c(0), c'(0)) \in T_x M = T_{c(0)} M$ and $c : (-\varepsilon, \varepsilon) \to M$

C^∞-curve.

<u>Lemma 3.3</u> The map $(*)$ is welldefined and linear.

<u>Proof.</u> Smoothness of $f \Rightarrow \exists$ open neighbhd. $U \subseteq \mathbb{R}^k$ of x
and C^∞-map. $\tilde{f} : U_x \to \mathbb{R}^m$ s.t. $\tilde{f}|_{M \cap U_x} = f|_{M \cap U_x}$.
We may assume that $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = x$ ~~sdisit~~
satisfies $c((-\varepsilon, \varepsilon)) \subset M \cap U_x$.

$\Rightarrow \tilde{f} \circ c = f \circ c : (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ smooth curve.

and $\underline{(f \circ c)'(0)} = (\tilde{f} \circ c)'(0) = D_{c(0)} \tilde{f} \, c'(0)$

$\Rightarrow$ $(*)$ is well-defined (just depends on $c'(0)$) and it is linear, since it is the restriction of a linear map $T_x\tilde{f} : T_x\mathbb{R}^n \to T_{\tilde{f}(x)}\mathbb{R}^m$ to $T_xM \subseteq T_x\mathbb{R}^n$,

__Def. 3.4__ $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ submflds., $f: M \to N$ C^∞-map. Then the tangent map of f at x is given by

$$T_xf : T_xM \longrightarrow T_{f(x)}N$$

$$T_xf\left(c(0), c'(0)\right) := \left(f(c(0)), (f\circ c)'(0)\right)$$

for a tangent vector $(x,v) = \left(c(0), c'(0)\right) \in T_xM$.

From the definition and chain rule in $\mathbb{R}^n$:

Cor. 3.5 If $f: M \to N$, $g: N \to P$ are smooth maps between submfd. $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ and $P \subseteq \mathbb{R}^e$.

1. $T_x(g \circ f) = T_{f(x)} g \circ T_x f : T_x M \to T_{g(f(x))} P$.

2. If $f: M \to N$ is a diffeomorphism, then for any $x \in M$ $T_x f : T_x M \to T_{f(x)} N$ is an isomorphism with inverse $(T_x f)^{-1} = T_{f(x)} f^{-1}$.

Proof

① We can find smooth extension $\tilde{f}$ and $\tilde{g}$ of f resp. g locally around x resp. $f(x)$ to open subsets of the ambient vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$.

$$\Longrightarrow \quad T_x(g\circ f) = \left(g(f(x)),\ D_x(\tilde{g}\circ\tilde{f})\big|_{T_xM}\right) = \left(g(f(x)),\ D_{\tilde{f}(x)}\tilde{g}\big|_{T_{f(x)}N} \circ D_x\tilde{f}\big|_{T_xM}\right)$$

$$= T_{f(x)}g \circ T_x f\,.$$

② We have $f^{-1}\circ f = \mathrm{Id}_M$ and $f\circ f^{-1} = \mathrm{id}_N$ and $T_x\,\mathrm{Id}_M = \mathrm{Id}\big|_{T_xM}$ $\forall x\in M$. By ①, $T_x(f^{-1}\circ f) = T_x\,\mathrm{Id}_M = \mathrm{Id}\big|_{T_xM}$

$$= T_{f(x)}f^{-1}\circ T_x f$$

and $\mathrm{Id}_{T_{f(x)}N} = T_{f(x)}(f \circ f^{-1}) \underset{①}{=} T_x f \circ T_{f(x)} f^{-1}$ . .

Cor. 3.6 (Inverse Function Thm.)

Let $f: M \to N$ be a smooth fct. between submfd. $M \subseteq \mathbb{R}^k$, $N \subseteq \mathbb{R}^{\ell}$.

① If for $x \in M$, $T_x f: T_x M \to T_{f(x)} N$ is a linear isomorphism.,
then $\exists$ open subsets W_1 of M and W_2 of N with
$x \in W_1$ and $f(x) \in W_2$ s.th $f|_{W_1}: W_1 \to W_2$ is a diffeom.

② $f: M \to N$ is a local diffeom. $\Longleftrightarrow$ $T_x f: T_x M \to T_{f(x)} N$
is a linear isomorphism $\forall x \in M$.

Proof

① Let (U, u) a chart of M with $x \in U$ and (V, v) a chart of N with $f(x) \in V$.

$\Longrightarrow \quad v \circ f \circ u^{-1} : u(U \cap f^{-1}(V)) \longrightarrow v(V)$ smooth

maps between open subset of $\mathbb{R}^k$.

We have $T_{u(x)}(v \circ f \circ u^{-1}) = T_{f(x)} v \circ T_x f \circ T_{u(x)} u^{-1}$, which

is as the composition of three linear isomorphisms an isomorphism.

By the inverse function Thm., $\exists$ open neighborhds. $\tilde{W}_1$ of $u(x)$

in $\mathbb{R}^k$ ~~and~~ s.t. $(v \circ f \circ u^{-1})|_{\tilde{W}_1} =: \tilde{W}_2$ is open and a smooth map $g : \tilde{W}_2 \to \tilde{W}_1$ inverse to $(v \circ f \circ u^{-1})|_{\tilde{W}_1}$.

Then $W_1 := u^{-1}(\tilde{W}_1)$ is open in M and $W_2 = f(W_1) = v^{-1}(\tilde{W}_2)$.
is open in N and $(u^{-1} \circ g \circ v)|_{W_2} : W_2 \to W_1$
is inverse to $f|_{W_1}$. $\quad \left(v^{-1} \circ \underbrace{v \circ f \circ u^{-1} \circ g}_{id} \circ v = v^{-1} \circ v = id \right)$.

② Follows directly from ① and Corollary 3.5.

3.2. The tangent bundle of a submfd.

Ordinary differ. eq. (of first order) on manifolds are described by vector fields on M. To be able to speak of smoothness of those, we need to give the disjoint union of all tangent spaces a smooth structure.

Recall: First order differential eq. is given by

$$x'(t) = f(x(t)) \qquad f : \underset{\subseteq \mathbb{R}^n \text{ open.}}{U} \longrightarrow \mathbb{R}^n \quad \text{smooth}$$

$\exists$ solution with initial cond. $x(0) = x_0$ given by a C^∞- curve $x : (a,b) \longrightarrow U$ with $x(0) = x$.

If we replace U by a submfd. M, then a solution

is a smooth curve $x : (a,b) \longrightarrow M$, which implies $x'(t) \in T_{x(t)}M$.

So f has to be a map $f : \underset{\in M}{x} \longmapsto f(x) \in T_x M$.

i.e. $f : M \longrightarrow \bigsqcup_{x \in M} T_x M$

$x \longmapsto f(x) \in T_x M$.

To speak about smoothness of f we need a C^∞-structure

on $\bigsqcup_{x \in M} T_x M$.

Def. 3.7 $M \subseteq \mathbb{R}^n$ submfd.

① • $TM := \bigsqcup_{x \in M} T_x M = \bigcup_{x \in M} \{x\} \times T_x M = \{(x, v) \in T_x M : x \in M\}$.

$\subseteq \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$.

• $p : TM \longrightarrow M$

$p((x, v)) = x$

TM is called the tangent space of M and $p : TM \to M$ is called the tangent bundle of M.

② If $N \subseteq \mathbb{R}^m$ is another submfd. and $f : M \longrightarrow N$ a smooth map, then the tangent map of f is

$Tf : TM \to TN$ is given by $Tf(x,v) = T_x f(x,v)$.

(often we also just write $v \in T_xM$ and $Tf(v) = T_x f v$.).

<u>Thm. 3.8</u> $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ and $P \subseteq \mathbb{R}^c$ submfds.

① $TM \subseteq \mathbb{R}^{2n}$ is a submf. of $\mathbb{R}^{2n}$ and $p : TM \to M$ is smooth. (If M has dim. k, then TM has dim. $2k$).

② For a smooth map $f : M \to N$, the tangent map $Tf : TM \to TN$ is smooth.

③ If $g : N \to P$ is another smooth map, then $T(g \circ f) = Tg \circ Tf$.

In particular, if f is a diffeom., then Tf is a diffeom. and $(Tf)^{-1} = Tf^{-1}$.

<u>Proof</u> A

① $x \in M$, $\psi : \tilde{U} \to \mathbb{R}^{n-k}$ regular and smooth. s.t.

$\tilde{U} \subseteq \mathbb{R}^n$ open neigh. of x

Assume $\dim(M) = k$.

$\psi^{-1}(0) = M \cap \tilde{U}$.

$\tilde{V} := \{ (y, v) \in \mathbb{R}^n \times \mathbb{R}^n : y \in \tilde{U} \} = \tilde{U} \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ open subset.

$\Psi : \tilde{V} \longrightarrow \mathbb{R}^{n-k} \times \mathbb{R}^{n-k}$

$\Psi(y, v) := (\psi(y), D_y \psi v)$ is smooth.

- $\underline{\Psi}(y,v) = 0 \iff y \in M$ and $\underline{D_y \varphi\, v = 0}$

$\underset{\text{Prop. 3.1.}}{\iff} (y,v) \in T_y M$.

i.e. $\underline{\underline{\Psi}}^{-1}(0,0) = TM \cap \tilde{V}$.

Regularity: $D_{(y,v)}\underline{\Psi} = \begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{n} \\ \left(\begin{array}{c} D_y \varphi \\ * \end{array}\right. & \left.\begin{array}{c} 0 \\ D_y \underline{\varphi} \end{array}\right) \end{array} \begin{array}{l} \}n-k \\ \}n-k \end{array}$

Regularity of φ implies that $D_{(y,v)}\underline{\Psi}$ is surjective onto $\mathbb{R}^{n-k} \times \mathbb{R}^{n-k}$.

$\Rightarrow TM \subseteq \mathbb{R}^{2n}$ is a $2k$-dim. submf. of $\mathbb{R}^{2n}$.

The projection $p_1 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a smooth extension of $p : TM \longrightarrow M$ and so ~~the~~ the latter is smooth.

② Smoothness of f implies: for $x \in M$ $\exists$ an open neighbhd.
$\tilde{U}_x \subseteq \mathbb{R}^n$ and $\tilde{f} : \tilde{U}_x \to \mathbb{R}^m$ C^∞-map s.t. $\tilde{f}\big|_{\tilde{U}_x \cap M} = f\big|_{\tilde{U}_x \cap M}$.

$$\tilde{V} := \{ (y,v) \in \mathbb{R}^n \times \mathbb{R}^n : y \in \tilde{U}_x \}.$$

$$F : \tilde{V} \longrightarrow \mathbb{R}^m \times \mathbb{R}^m$$

$$F(y,v) := \left(\tilde{f}(y), D_y \tilde{f}\, v \right).$$ is smooth

- for $(y,v) \in \tilde{V} \cap TM$ we have $f(y) = \tilde{f}(y)$ and

$$F(y,v) = Tf(y,v) = T_y f(y,v).$$

$\Longrightarrow$ F is a smooth extension of Tf and so Tf is smooth.

③ $T(g\circ f)(x,v) = T_x(g\circ f)(x,v) \underset{\text{Cor. 3.5}}{=} T_{f(x)}g \circ T_x f(x,v) = Tg \circ Tf(x,v)$

This also implies statement about diffeom. as in Cor. 3.5.

Distinguished charts for TM: $M \subseteq \mathbb{R}^n$ subpf. of dim. k.

(U,u) a chart for M: $u : \underset{\substack{\subseteq M \\ \text{open subset}}}{U} \longrightarrow u(U) \subseteq \mathbb{R}^k$ diffeom.

- $Tu(U) = u(U) \times \mathbb{R}^k \subseteq \mathbb{R}^k \times \mathbb{R}^k = \mathbb{R}^{2k}$.
- $TU = p^{-1}(U) \subseteq TM$ is open, since p is continuous.
- $Tu : TU \longrightarrow T(u(U))$ is a diffeom. by ③ of Thm. 3.8.

Hence, (TU, Tu) is a chart for TM.

Suppose (U_α, u_α) and (U_β, u_β) two charts for M with $U_\alpha \cap U_\beta \neq \emptyset$:

$$u_{\beta\alpha} := u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to u_\beta(U_\alpha \cap U_\beta)$$

diffeom. between open subsets of $\mathbb{R}^k$.

- $p^{-1}(U_\alpha) \cap p^{-1}(U_\beta) = p^{-1}(U_\alpha \cap U_\beta) = T(U_\alpha \cap U_\beta) \subseteq TM$ open subset.
- $Tu_\beta \circ (Tu_\alpha)^{-1} = T(u_\beta \circ u_\alpha^{-1}) : T(u_\alpha(U_\alpha \cap U_\beta)) \to T(u_\beta(U_\alpha \cap U_\beta))$

which equals $(y,v) \longmapsto \left(\underline{u_\beta \circ u_\alpha^{-1}(y)}, \underline{D_y(u_\beta \circ u_\alpha^{-1}) v} \right)$.

Hence, transition maps on TM coincide with the transition maps of and their derivatives.

($\Rightarrow$ Atlas on M gives rise to atlas on TM).