Yesterday: We defined the tangent space TM of a submfd $M \subseteq \mathbb{R}^n$.

$$TM = \bigsqcup_{x \in M} T_x M = \underbrace{\bigcup_{x\in M} \{x\} \times T_x M}_{(x,x,v)} = \{ (x,v) \in T_x M : x \in M \}$$

...

↑ union of the $T_x M$'s

$T_x M \subseteq \mathbb{R}^n$ ~~...~~, i.e. $T_x M$ is identified with subspace of $\mathbb{R}^n$

$f : M \to N$ smooth map between submfds $M \subseteq \mathbb{R}^n$ and $N \subseteq \mathbb{R}^m$ of dim k resp. ℓ.

Fix $x \in M$, let (U,u) be a chart on M with $x \in U$ and (V,v) a chart on N with $f(x) \in V$.

$$v \circ f \circ u^{-1} : \underbrace{u(U \cap f^{-1}(V))}_{\subseteq \mathbb{R}^k} \longrightarrow \underbrace{v(V)}_{\subseteq \mathbb{R}^\ell}$$

$(f^1, \ldots, f^\ell)$ local coordinate expression of f w.r.t. (U,u) and (V,v).

$$f^i : u(U \cap f^{-1}(V)) \longrightarrow \mathbb{R}.$$

- (TU, T_u), (TV, T_v) cont for TM and TN.

$$T_v \circ Tf \circ T_u^{-1} = T(v \circ f \circ u^{-1}) : T\big(u(U \cap f^{-1}(V))\big) \longrightarrow T(v(V))$$

$$(y, v^1, \dots, v^k)$$

$$y \in u(U \cap f^{-1}(V)) \longmapsto \Big(f^1(y), \dots, f^l(y), \frac{\partial f^1}{\partial x^1}(y) v^1 + \dots + \frac{\partial f^1}{\partial x^k}(y) v^k, \dots$$

$$\dots \frac{\partial f^l}{\partial x^1}(y) v^1 + \dots + \frac{\partial f^l}{\partial x^k}(y) v^k\Big)$$

$$= \left(f^1(y), \dots, f^l(y), D_y(f^1, \dots, f^l)\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}\right).$$

3.3 Vector fields

Def. 3.9 M is a manifold. A (smooth) vector bundle of rank k over M is a manifold E together with a smooth surjective map $p: E \to M$ s.t.:

- for $x \in M$, $p^{-1}(x) =: E_x$ (called the fiber of p over x). is endowed with structure of a real k-dim. vector space.
- for any $x \in M$ $\exists$ an open subset U of x in M and a diffeom. $\Phi: p^{-1}(U) \to U \times \mathbb{R}^k$ s.t. the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & U \times \mathbb{R}^k \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

$pr_1 : U \times \mathbb{R}^k \to U$
natural projection.

and $\Phi\big|_{p^{-1}(y)} : E_y \longrightarrow \{y\} \times \mathbb{R}^k \simeq \mathbb{R}^k$ is a linear isomorphism $\forall y \in U$

- Φ is called a local trivialization of E
- E is the total space of the vector bundle $p : E \to M$ and M the base.

Ex. M is a manifold, then $M \times \mathbb{R}^k$ is a mfd. and

$pr_1 : M \times \mathbb{R}^k \longrightarrow M$ is a vector bundle of rank k over M.

(Trivial vector bundle over M).

Ex. $M \subseteq \mathbb{R}^n$ submfd. of dim. k

Then $p : TM \longrightarrow M$ is a vector bundle of rank k over M,

called the tangent bundle of M.

Take a chart (U,u) for M with $x \in \cancel{M}\, U$, then

$$Tu : TU \longrightarrow \underline{\underline{u(U)}} \times \mathbb{R}^k \subseteq \mathbb{R}^k \times \mathbb{R}^k$$

$$\begin{tikzcd}
p^{-1}(U) = TU \arrow[r, "Tu"] \arrow[dr, bend right, "p"'] \arrow[rr, bend left, "\phi"] & u(U) \times \mathbb{R}^k \arrow[r, "u^{-1} \times id"] & U \times \mathbb{R}^k \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}$$

$(y,v) \in T_y U$.

- $\phi = (u^{-1} \times id) \circ T_U$ is a diffeomorphism and $pr_1(\phi(y,v)) = y$
  $= p(y,v)$.
- $\phi\big|_{T_y U} : T_y U \longrightarrow \{y\} \times \mathbb{R}^k \simeq \mathbb{R}^k$

equals $\underbrace{T_y U \xrightarrow[\sim]{T_y u} T_{u(y)} u(U) \simeq \mathbb{R}^k}_{= \phi|_{T_y U}}$, which is a linear isomorphism.

Remark If $p: E \to M$ is a vector bundle over a mf. M and $U \subseteq M$ an open subset, then $p^{-1}(U) =: E_U \xrightarrow{p} U$ is a vector bundle over U.

Def. 3.10 $p: E \to M$ is a vector bundle over a mfd. M.

- A (smooth) section of p is a smooth map $s: M \to E$ s.t. $p \circ s = \mathrm{id}_M$ (i.e. $s(x) \in E_x \ \forall x \in M$).
- If $U \subseteq M$ is open, then a section of $E|_U$ is called a local section of p defined on U.

Def. 3.11 Two vector bundles $p_1 : E^1 \to M$ and $p_2 : E^2 \to M$ vector bundle are isomorphic, if $\exists$ a diffeom. $F : E^1 \to E^2$ s.t. the following diagram commutes

$$\begin{tikzcd} E^1 \arrow[r, "f"] \arrow[dr, "p_1"'] & E^2 \arrow[d, "p_2"] \\ & M \end{tikzcd}$$

and $F|_{E^1_x} : E^1_x \simeq E^2_x$ is a linear isomorphism.

Def. 3.12 $M \subseteq \mathbb{R}^n$ submfld.

- Then a (smooth) vector field on M is a (smooth) section $\xi : M \to TM$ of $p : TM \to M$.

- A local vector field defined on open subset $U \subseteq M$ is a section $s : U \to TU$ of $TM|_U = p^{-1}(U) = TU$.

Notation: • $\xi(x) = (x, \xi_x) \in T_x M$; sometimes we will just identify ξ_x with $\xi(x)$.

($s(x) = (x, 0) \in T_x M$ $\longrightarrow$ zero section of TM.)

• $\mathfrak{X}(M)$ or $\Gamma(TM)$ denotes the set of all vector fields on M.

Def. 3.13 $\xi \in \mathfrak{X}(M)$

• $\mathrm{supp}(\xi) = \overline{\{x \in M : \xi_x \neq 0\}}$ support of ξ.

Lemma 3.14 $\mathfrak{X}(M) = \Gamma(TM)$ is a vector space:

$\xi, \eta \in \mathfrak{X}(M)$ $\quad (\xi + \eta)(x) := \xi(x) + \eta(x)$

$\lambda \in \mathbb{R}$ $\quad (\lambda \xi)(x) := \lambda \xi(x)$

Moreover, it is a module over the ring $C^\infty(M, \mathbb{R})$: $f \in C^\infty(M, \mathbb{R})$

$$(f\xi)(x) := f(x)\xi(x).$$

Ex. $M \subseteq \mathbb{R}^n$ submfd, $\dim(M) = k$.

Let (U, u) be a chart for M and (TU, Tu) corresp. chart for TM.

$$\begin{tikzcd} TU \arrow[rr, "\sim", "\phi"'] \arrow[dr, "p"'] & & U \times \mathbb{R}^k \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

$\phi = (u^{-1} \times id) \circ Tu$.

$y \in U$: $\frac{\partial}{\partial u^i}(y) = \phi^{-1}(y, e^i)$ e^i i-the standard basis vector of $\mathbb{R}^k$

$\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^k}$ local vector fields defined on U.

$\left(Tu \circ \frac{\partial}{\partial u^i} \circ u^{-1} : u(U) \to u(U) \times \mathbb{R}^k \text{ is smooth} \right)$.

$$\begin{aligned} &\quad\; \rotatebox{90}{$\in$} \\ z &\longmapsto (z, e^i) \end{aligned}$$

Coordinate vector fields associated to (U, u).

For $y \in U$, $\frac{\partial}{\partial u^1}(y), \dots, \frac{\partial}{\partial u^k}(y)$ is a basis of $T_y U = T_y M$.

If $\xi^i \in C^\infty(U,\mathbb{R})$ C^∞-fcts. $i=1,\dots,k$, then by Lemma 3.14

$$\xi = \sum_{i=1}^{k} \xi^i \frac{\partial}{\partial u^i}$$ is a vector field over U.

($\Rightarrow$ ~~$\exists$~~ locally $\exists$ many vector fields on a smooth manifold).

Conversely, if $\xi \in \mathfrak{X}(U)$, then for $y \in U$ we may write

$$\xi(y) = \sum_{i=1}^{k} \xi^i(y) \frac{\partial}{\partial u^i}(y) \quad \text{for } \xi^i(y) \in \mathbb{R}.$$

$\xi^i : U \longrightarrow \mathbb{R}$ are smooth: $Tu \circ \xi \circ u^{-1} : u(U) \to u(U) \times \mathbb{R}^k$

$(u^1(y), \dots, u^k(y)) \longmapsto (u^1(y), \dots, u^k(y), \xi^1(y), \dots, \xi^k(y))$

Moreover, if $\xi \in \Gamma(TU)$, $x \in U$, then $\exists$ an open neighborhood V of x in M s.t. $\overline{V} \subseteq U$. By Cor. 2.32, $\exists$ a smooth fct $\phi : M \to \mathbb{R}$ s.t. $\mathrm{supp}(\phi) \subset U$ and $\phi|_{\overline{V}} \equiv 1$.

$$\tilde{\xi}(y) := \begin{cases} \phi(y)\,\xi(y) & y \in U \\ 0 & y \in M \setminus U \end{cases}$$

$\Rightarrow$ $\tilde{\xi} \in \mathfrak{X}(M)$ and $\tilde{\xi}|_V = \xi|_V$.

$\Rightarrow$ M has many globally defined vector fields.

Def. 3.15 $M \subseteq \mathbb{R}^n$ submfd., $\xi \in \mathfrak{X}(M)$ and (U, u) a coord chart.

$$\xi\big|_U \in \mathfrak{X}(U) \quad \text{and} \quad \xi\big|_U = \sum_{i=1}^{k} \xi^i \frac{\partial}{\partial u^i} \quad \text{for } \xi^i \in C^\infty(U, \mathbb{R}).$$

$(\xi^1, \dots, \xi^k)$ is called the local coordinate expression of ξ with respect to (U, u).

(or $\xi^i \circ u^{-1} : u(U) \to \mathbb{R}$ is called like that).

Suppose $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ is an atlas for $M \subseteq \mathbb{R}^n$.

(U_α, u_α), (U_β, u_β) with $U_\alpha \cap U_\beta \neq \emptyset$.
$\qquad =: U_{\alpha\beta}$

$$u_{\beta\alpha} := u_\beta \circ u_\alpha^{-1} : \underbrace{u_\alpha(U_{\alpha\beta})}_{\subseteq \mathbb{R}^k} \to \underbrace{u_\beta(U_{\alpha\beta})}_{\subseteq \mathbb{R}^k}.$$

$$T u_\beta \circ T u_\alpha^{-1} : \overbrace{T u_\alpha(U_{\alpha\beta})}^{} \to T u_\beta(U_{\alpha\beta})$$

$$\overset{\cong u_\alpha(U_{\alpha\beta}) \times \mathbb{R}^k}{(y, v)} \longmapsto \left(u_{\beta\alpha}(y), D_y u_{\beta\alpha} v \right).$$

For $x \in U_{\alpha\beta}$ set $A^i_j(x) := \dfrac{\partial u^i_{\alpha\beta}}{\partial y^j}(u_\alpha(x))$.

Then $U_{\alpha\beta} \longrightarrow GL(k, \mathbb{R})$ is smooth.

$x \longmapsto \{A^i_j(x)\}$

- $\frac{\partial}{\partial u^i_\alpha}(x) = Tu_\alpha^{-1}\left(u_\alpha(x), e^i\right)$

- $T_x U_\beta\left(\frac{\partial}{\partial u^i_\alpha}(x)\right) = \left(u_\beta(x),\ \underbrace{D_{u_\alpha(x)} u_{\beta\alpha} e^i}_{}\right)$

$\uparrow$ = i-th column of A.

$$\Longrightarrow \frac{\partial}{\partial u^i_\alpha}(x) = \sum_{j=1}^{k} A^j_i \frac{\partial}{\partial u^j_\beta}.$$

Suppose $\xi \in \mathfrak{X}(M)$ and $(\xi_\alpha^1, \dots, \xi_\alpha^k)$ and $(\xi_\beta^1, \dots, \xi_\beta^k)$ are the local coordinate expressions of $\xi|_{U_{\alpha\beta}}$ with respect to (U_α, u_α) and (U_β, u_β) then on $U_{\alpha\beta}$:

$$\xi|_{U_{\alpha\beta}} = \sum_i \xi_\alpha^i \underline{\frac{\partial}{\partial u_\alpha^i}} = \sum_{i,j} \underbrace{\xi_\alpha^i A_i^j}_{= \xi_\beta^j} \frac{\partial}{\partial u_\beta^j} = \sum_j \xi_\beta^j \frac{\partial}{\partial u_\beta^j}$$

Ex. $M = \mathbb{R}^2$ $\qquad u_\beta = id_{\mathbb{R}^2}$ standard coordinates

$$u_\beta^1 = x^1 \qquad \longrightarrow \quad \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$$
$$u_\beta^2 = x^2,$$

$U_\alpha = \mathbb{R}^2 \setminus \{0\}$ $\quad u_\alpha$ ... polar coordinates (r, φ)

Jacobian of $id \circ u_\alpha^{-1} = u_\alpha^{-1}$ $\qquad \frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}$

$$u_\alpha^{-1}(r, \varphi) = (r\cos\varphi, r\sin\varphi).$$

$$D \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$\frac{\partial}{\partial r} = \cos\varphi \frac{\partial}{\partial x^1} + \sin\varphi \frac{\partial}{\partial x^2}$$
$$= \frac{1}{r}\left(x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}\right).$$

$$\frac{\partial}{\partial \varphi} = -r \sin\varphi \frac{\partial}{\partial x^1} + r\cos\varphi \frac{\partial}{\partial x^2} = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$$

<u>Def. 3.16</u> $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ submfds., $f : M \to N$ is a local diffeomorphism. For any vf $\eta \in \mathfrak{X}(N)$

$$f^*\eta : M \to TM$$
$$x \mapsto (T_x f)^{-1} \eta(f(x))$$

defines a vector field on M, called the pull-back of η w.r.t. to f.

Prop. 3.17 $f: M \to N$ local diffeom. between submflds of $\mathbb{R}^n$ and $\mathbb{R}^m$.

① $f^*: \mathfrak{X}(N) \longrightarrow \mathfrak{X}(M)$ is $\mathbb{R}$-linear and

for $h \in C^\infty(N, \mathbb{R})$ we have $f^*(h\eta) = (h \circ f) f^*\eta \quad \forall \eta \in \mathfrak{X}(N)$.

② $g: N \to P$ another local diffeom. between submflds, then

$$(g \circ f)^* \eta = f^*(g^* \eta).$$

$\eta \in \mathfrak{X}(P)$

③ $\mathrm{Id}_N^* \eta = \eta$ and if $M = U \subseteq N$ open subset

and $f = i: U \hookrightarrow N$ inclusion, then $i^* \eta = \eta|_U$.

Ex $M \subseteq \mathbb{R}^n$ subuf. , (U,u) chart for M.

- $u(U) \subseteq \mathbb{R}^k$ open subset and $\frac{\partial}{\partial x^i} : u(U) \to T_{u(U)} = u(U) \times \mathbb{R}^k$, $x \longmapsto (x, e^i)$.

  is a vector field on $u(U)$.

- $u : U \to u(U)$

$$u^* \frac{\partial}{\partial x^i}(y) = (T_y u)^{-1} \underbrace{\frac{\partial}{\partial x^i}(u(y))}_{(u(y), e^i)}$$

$$= (T_y u)^{-1}(u(y), e^i)$$

$$= \frac{\partial}{\partial u^i}.$$

In particular, $\xi \in \mathfrak{X}(M)$

$\Rightarrow$ $\xi|_U = \sum \xi^i \frac{\partial}{\partial u^i}$.

$$(u^{-1})^* \xi|_U = \sum_i \xi^i \circ u^{-1} \frac{\partial}{\partial x^i}.$$