Yesterday: $M \subseteq \mathbb{R}^n$ submfd., $x \in M$.

$$\varphi_x : T_x M \longrightarrow \mathrm{Der}_x(C^\infty(M,\mathbb{R}), \mathbb{R})$$

Claim: φ_x is a linear isomorphism.

~~Step~~ The rest of the proof (of Thm. 3.24); namely, we finish the proof that φ_x is surjective.

$f \in C^\infty(M,\mathbb{R})$, chart around x (U,u) with $u(x) = 0$ and $B_1(0) \subseteq u(U)$.

We have seen that for $y \in U$ with $u(y) \in B_1(0)$ we have

$$f(y) = f(x) + \sum_i u^i(y)\, h_i(y) \qquad h_i : u^{-1}(B_1(0)) \longrightarrow \mathbb{R}.$$

By Cor. 2.32 we can extend h_i and u^i to smooth fcts. on M without changing them locally around x: The function

$$f(x) + \sum_i u^i h_i \qquad (*)$$

is an be extended to fct. on M that coincides with f locally around x.

If $\partial \in Der_x(C^\infty(M,\mathbb{R}),\mathbb{R})$, then by Lemma 3.23:

$$\partial(f) = \partial\Big(f(x) + \sum_i u^i h_i\Big) = \sum_j \partial(u^i)\, \underset{\underset{\frac{\partial f}{\partial u^i}(x)}{\parallel}}{h_i(x)} + \underbrace{u^i(x)}_{=0}\, \partial(h_i)$$

$$= \sum_i \partial(u^i) \frac{\partial f}{\partial u^i}(x) \implies \partial = \partial_\xi \qquad \xi = \sum \partial(u^i) \frac{\partial}{\partial u^i}(x). \ \square$$

3.5 Tangent bundle (and tangent maps) of abstract manifolds

Suppose $(M, \mathcal{A})$ abstract mfd. of dim. k.

Then we define the tangent space of M at x as the vector space:

$$T_x M := \mathrm{Der}_x\left(C^\infty(M,\mathbb{R}),\mathbb{R}\right).$$

<u>Notation</u>: $\xi_x(f) := \xi_x \cdot f \quad \forall f \in C^\infty(M,\mathbb{R})$.

<u>Remark</u>: Alternatively, we could have defined $T_x M$ as the set of equivalence classes of smooth curves $c: I \to M$, $0 \in I$ ~~and~~ w.h. $c(0) = x$, where $c_1 \sim c_2$, if $x = c_1(0) = c_2(0)$ and for a (equiv., any) chart (U,u) around x $(u \circ c_1)'(0) = (u \circ c_2)'(0)$.

The tangent bundle of M is defined as

$$TM := \bigsqcup_{x\in M} T_xM = \bigcup_{x\in M} \{x\} \times T_xM$$

$p : TM \longrightarrow M$ natural projection.

For a smooth map between mfds. $f : M \longrightarrow N$ we define

$$Tf(x, \xi_x) := (f(x), T_xf\xi_x)$$

(we sometimes just write $Tf(x,\xi_x) = T_xf\xi_x$).

where $T_xf : T_xM \longrightarrow T_{f(x)}N$ is given by

$$T_xf(\xi_x)(g) := (T_xf\xi_x)\cdot g := \xi_x(g\circ f) =: \xi_x\cdot(g\circ f)$$

Thm. 3.24.

$\forall\, g \in C^\infty(N, \mathbb{R})$.

One verifies directly that:

- $T(h \circ f) = Th \circ Tf$ for $h: N \to P$ C^∞-map between mfds.
- $T\mathrm{Id}_M = \mathrm{Id}_{TM}$
- f is a local diffeom. $\Longleftrightarrow$ $T_x f : T_x M \to T_{f(x)} N$ is a linear isom. $\forall x \in M$.

$(u, U) \in \mathcal{A}$ $\qquad T_u : TU = p^{-1}(U) \longrightarrow Tu(U) = u(U) \times \mathbb{R}^k$.

There exists a unique topology on TM s.t. $TU \subset TM$ is open and $T_u : TU \to Tu(U)$ is a homeomorphism $\forall (U, u) \in \mathcal{A}$.
It is second countable and Hausdorff. Moreover, $\mathcal{A}_{TM} := \{(TU, T_u) : (U, u) \in \mathcal{A}\}$

defines a C^∞-atlas of charts with values in $\mathbb{R}^{2k}$ (see the corresp. statement for submfd $M \subseteq \mathbb{R}^n$).

$\Longrightarrow$ $(TM, \mathcal{A}_{TM})$ is a smooth manifold of dim $2k$.

Moreover, as for submfd. of $\mathbb{R}^n$, $p: TM \to M$ is smooth and it defines a vector bundle of rank k over M, and vector fields on M are defined as (smooth) sections of $p: TM \to M$.

Local coordinate expressions for the tangent map Tf of a smooth map $f: M \to N$, which is again smooth, and ~~la~~ for vector fields remain valid.

Definitions | Statements about pull-back of vector fields via local diffeom. and local flows of vector fields remain valid without any change.

3.6 Vector fields as derivations and the Lie bracket

$(M, \mathcal{A})$ a manifold.

For $\xi \in \mathfrak{X}(M)$ and $f \in C^\infty(M, \mathbb{R})$

$$\xi \cdot f : M \longrightarrow \mathbb{R}$$
$$(\xi \cdot f)(x) := \xi_x \cdot f = T_x f \xi_x$$

defines a smooth fct, since $\xi \cdot f$ is the second component of $Tf \circ \xi : M \longrightarrow TM \longrightarrow T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$, which is smooth.

Def. 3.25 A derivation of the algebra $C^\infty(M,\mathbb{R})$ is a linear map $D: C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ s.t. $D(fg) = D(f)g + fD(g)$ $\forall f,g \in C^\infty(M,\mathbb{R})$.

Notation: $\mathrm{Der}(C^\infty(M,\mathbb{R})) := \{ D : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}) : D \text{ is a derivation} \}$.

This is a vector space in the obvious way.

Thm 3.26 The map $\Psi : \xi \longmapsto (f \longmapsto \xi \cdot f)$ defines a linear isomorphism $\mathfrak{X}(M) \xrightarrow{\sim} \mathrm{Der}(C^\infty(M,\mathbb{R}))$.

Proof: $f \longmapsto \xi \cdot f$ linear $C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ ✓

$$\xi \cdot (fg)(x) = \xi_x \cdot (fg) = (\xi_x \cdot f)\, g(x) + f(x)\,(\xi_x \cdot g) = \big((\xi \cdot f)g + f(\xi \cdot g)\big)(x)$$

So Ψ has image in $\mathrm{Der}(C^\infty(M,\mathbb{R}))$.

Hence, φ defines a map $\mathfrak{X}(M) \to \mathrm{Der}(C^\infty(M,\mathbb{R}))$ and evidently it is linear, since $T_x f$ is linear $\forall x \in M$ and $f \in C^\infty(M,\mathbb{R})$.

Injectivity of φ: If $\xi \neq 0$, then $\exists x \in M$ s.t. $\underset{= \xi_x}{\xi(x)} \neq 0$

By Thm 3.24, we know that $\exists f \in C^\infty(M,\mathbb{R})$ s.t.
$\xi_x \cdot f = (\xi \cdot f)(x) \neq 0$.

Surjectivity of φ: Let $D \in \mathrm{Der}(C^\infty(M,\mathbb{R}))$. For any $x \in M$
$f \mapsto D(f)(x)$ is a derivation at x. Hence, by Thm. 3.24
$\exists!\, \xi_x \in T_xM$ s.t. $D(f)(x) = \xi_x \cdot f$. Remains to show that
$x \mapsto \xi_x$ defines a vector field, i.e it remains to show smoothness.

Fix $x \in M$ and a chart (U, u) with $x \in U$.

As in the proof of Thm. 3.24, we may extend u^i $(i = 1, \dots, k)$ to a smooth fct $\tilde{u}^i : M \to \mathbb{R}$ that coincide with u^i on some open neighbh. $V \subset U$ of x.

Then $D(\tilde{u}^i) : M \to \mathbb{R}$ is a smooth fct and

$$\xi_y = \sum_i (\xi_y \cdot \tilde{u}^i) \frac{\partial}{\partial u^i}(y) \quad \forall y \in V$$

(see proof of Thm. 3.24).

Hence, $\xi|_V = \sum_{i=1}^{k} D(\tilde{u}^i)|_V \frac{\partial}{\partial u^i}$ is a smooth vector field on V.

$\square$

Recall that for a chart (U,u), $\frac{\partial}{\partial u^i} \cdot f = \frac{\partial f}{\partial u^i}$

equals the i-th partial deriv. of local coordinate expression $f \circ u^{-1}$ of f. This implies that for any $\xi \in \mathfrak{X}(M)$ with $\xi|_U = \sum \xi^i \frac{\partial}{\partial u^i}$ we have $(\xi \cdot f)|_U = \sum_i \xi^i \frac{\partial f}{\partial u^i}$.

<u>Lemma 3.27</u> $\xi, \eta \in \mathfrak{X}(M)$ vector fields on a mfd. M.
Then $f \longmapsto (\underline{\xi \cdot (\eta \cdot f)}) - \eta \cdot (\xi \cdot f)$ defines a derivation of $C^\infty(M,\mathbb{R})$.

<u>Proof</u> $f, g \in C^\infty(M,\mathbb{R})$

$$\xi \cdot (\eta \cdot (fg)) = \xi \cdot \{(\underline{\eta \cdot f}) g + f(\eta \cdot g)\} = \overline{(\xi \cdot (\eta \cdot f))} g + (\eta \cdot f)(\xi \cdot g) + (\xi \cdot f)(\eta \cdot g) + \underline{f(\xi \cdot (\eta \cdot g))} \quad \square$$

Def. 3.28 M mfd. For two vector fields $\xi, \eta \in \mathfrak{X}(M)$ the Lie bracket of ξ and η is the unique vector fields $[\xi, \eta] \in \mathfrak{X}(M)$ s.t. $[\xi, \eta] \cdot f = \xi \cdot (\eta \cdot f) - \eta \cdot (\xi \cdot f) \quad \forall f \in C^\infty(M, \mathbb{R})$.

Prop. 3.29 M mfd., $\xi, \eta, \varrho \in \mathfrak{X}(M)$.

1. $[\xi, \eta] = -[\eta, \xi]$ and $[\xi, [\eta, \varrho]] + [\eta, [\varrho, \xi]] + [\varrho, [\xi, \eta]] = 0$ (Jacobi identity).

2. $[\xi, f\eta] = f[\xi, \eta] + (\xi \cdot f)\eta$

   and $[f\xi, \eta] = f[\xi, \eta] - (\eta \cdot f)\xi$.

Proof.

(1) Skew-symmetry ✓ and Jacobi identity follows from mindless computations.

(2) $f, g \in C^\infty(M, \mathbb{R})$ $\quad [s, f\eta] \cdot g = \dots$ ✓

$$((f\eta) \cdot g)(x) = f(x)\, \eta_x \cdot g = (f(\eta \cdot g))(x) \quad \dots$$

$$\Longrightarrow s \cdot (\underline{(f\eta) \cdot g}) = s \cdot (f(\eta \cdot g)) = \underline{(s \cdot f)(\eta \cdot g)} + \underline{f(s \cdot (\eta \cdot g))} \quad (*)$$

$$(f\eta) \cdot (s \cdot g) = \underline{f(\eta \cdot (s \cdot g))} \quad (**)$$

$$\Longrightarrow [s, f\eta] \cdot g = (*) - (**) = f([s,\eta] \cdot g) + (s \cdot f)\, \eta \cdot g \, .$$

$$\Longrightarrow [s, f\eta] = f[s, \eta] + (s \cdot f)\, \eta \; ✓ .$$

Prop. 3.30 M, N mfds, $f: M \to N$ a local diffeomorphism.

① $f^*[\xi, \eta] = [f^*\xi, f^*\eta] \quad \forall \xi, \eta \in \mathfrak{X}(N)$.

In particular, if $U \subseteq N$ is an open subset, $[\xi,\eta]|_U = [\xi|_U, \eta|_U]$

$(i: U \xrightarrow{\sim} i(U) \subseteq N$ ~~diffeom.~~

inclusion is diffeom. onto its image; $i^*\xi = \xi|_U$.

Hence, $\xi|_U \equiv 0$ implies $[\xi,\eta]|_U = 0$.

② $\frac{d}{dt}\Big|_{t=0} \underbrace{\left(Fl_t^{\xi *}\eta\right)(x)}_{\in T_x M} = [\xi,\eta](x) \quad \forall \xi,\eta \in \mathfrak{X}(M), x \in M$.

3 Suppose (U, u) is a chart on M and $\xi, \eta \in \mathfrak{X}(M)$ with

$\xi|_U = \sum \xi^i \frac{\partial}{\partial u^i}$ and $\eta|_U = \sum \eta^i \frac{\partial}{\partial u^i}$, then

$$[\xi, \eta]\big|_U = \sum_{i=1}^{n} [\xi, \eta]^i \frac{\partial}{\partial u^i},$$

where
$$[\xi, \eta]^i = \sum_{j=1}^{n} \left(\xi^j \frac{\partial \eta^i}{\partial u^j} - \eta^j \frac{\partial \xi^i}{\partial u^j} \right).$$