Prop. 3.30 M, N mfds, $f: M \to N$ a local diffeom.

① $f^*[\xi, \eta] = [f^*\xi, f^*\eta] \quad \forall \xi, \eta \in \mathfrak{X}(N)$.

In particular, if $U \subseteq M$ is an open subset and $i: U \hookrightarrow M$ inclusion, then $[\xi, \eta]|_U = i^*[\xi, \eta] = [i^*\xi, i^*\eta] = [\xi|_U, \eta|_U] \quad \forall \xi, \eta \in \mathfrak{X}(M)$.

Hence, $\xi|_U \equiv 0$ implies $[\xi, \eta]|_U \equiv 0 \quad \forall \eta \in \mathfrak{X}(M)$.

② For $\xi, \eta \in \mathfrak{X}(M)$, $x \in M$: $\left.\frac{d}{dt}\right|_{t=0} Fl_t^{\xi *}\eta(x) = [\xi, \eta](x)$.
$\in T_xM$

③ Suppose (U, u) is a chart on M and $\xi, \eta \in \mathfrak{X}(M)$ with $\xi|_U = \sum_i \xi^i \frac{\partial}{\partial u^i}$ and $\eta|_U = \sum_i \eta^i \frac{\partial}{\partial u^i}$. Then

$$[\xi,\eta]\big|_U = \sum_i [\xi,\eta]^i \frac{\partial}{\partial u^i} \quad , \text{ where}$$

$$[\xi,\eta]^i = \sum_j \left(\xi^j \frac{\partial \eta^i}{\partial u^j} - \eta^j \frac{\partial \xi^i}{\partial u^j} \right) .$$

<u>Proof</u>

① $f^*\xi = \underline{(Tf)^{-1} \circ \xi \circ f} \qquad \xi \in \mathfrak{X}(N) \qquad f : M \to N$

$g \in C^\infty(N, \mathbb{R})$.

$$\left(f^*\xi \cdot (g \circ f) \right)(x) = \underline{(f^*\xi)_x} \cdot (g \circ f) = \left(T_x f \underline{(f^*\xi)_x} \right) \cdot g = \xi_{f(x)} \cdot g$$

i.e. $f^*\xi \cdot (g \circ f) = (\xi \cdot g) \circ f$

$$\underline{[f^*\xi, f^*\eta]}\cdot(g\circ f) = f^*_\xi\cdot(\underbrace{f^*_\eta\cdot(g\circ f)}_{(\eta\cdot g)\circ f}) - f^*_\eta\cdot(f^*_\xi\cdot(g\circ f))$$

$$\underbrace{}$$

$$= \xi\cdot(\eta\cdot g)\circ f - \eta\cdot(\xi\cdot g)\circ f$$

$$= ([\xi,\eta]\cdot g)\circ f = \underline{(f^*[\xi,\eta])}\cdot(g\circ f) .$$

③ By ①, $[\xi,\eta]|_U = [\xi|_U, \eta|_U] =$

$$= \sum_{i,j}[\xi^i\frac{\partial}{\partial u^i}, \underline{\eta^j}\frac{\partial}{\partial u^j}] \underset{\text{Prop. 3.28}}{=} \sum_{i,j}\left(\eta^j[\underline{\xi^i}\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] + \xi^i\frac{\partial\eta^j}{\partial u^i}\frac{\partial}{\partial u^j}\right)$$

$$\underset{\text{Prop-3.28}}{=} \sum_{i,j}\left(\xi^i\frac{\partial\eta^j}{\partial u^i}\frac{\partial}{\partial u^j} - \eta^j\frac{\partial\xi^i}{\partial u^j}\frac{\partial}{\partial u^i}\right), \text{ since}$$

$$\left[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right] = 0$$

Note that $\left[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right] \cdot f = \underbrace{\frac{\partial}{\partial u^i} \cdot \Big(\underbrace{\frac{\partial}{\partial u^j} \cdot f}_{\substack{j\text{-th} \\ \text{deriv. of} \\ f \circ u^{-1}}} \Big)}_{\frac{\partial^2 f}{\partial u^i \partial u^j}} - \frac{\partial}{\partial u^j} \cdot \left(\frac{\partial}{\partial u^i} \cdot f \right)$

$\frac{\partial^2 f}{\partial u^i \partial u^j}$ 2nd - partial derivative of $f \circ u^{-1}$.

$= 0$ by symmetry of 2nd partial derivatives.

② $\frac{d}{dt}\Big|_{t=0} \underline{(Fl_t^{\xi *})\eta}(x) = [\xi, \eta](x)$.

$t \mapsto (\underline{TFl_{-t}^{\xi}} \circ \eta \circ Fl_t^{\xi})(x)$ locally defined curve in $T_x M$.

Consider $\alpha : \mathbb{R}^2 \to \mathbb{R}$ given by $\qquad f \in C^\infty(M, \mathbb{R})$

$$\alpha(t,s) := \eta(Fl_t^{\xi}(x)) \cdot (f \circ Fl_s^{\xi}) =$$

$$= TFl_s^{\xi}(\eta(Fl_t^{\xi}(x)) \cdot f) = (TFl_s^{\xi} \circ \eta \circ Fl_t^{\xi})_x \cdot f$$

$\cdot\ \alpha(t,0) = \eta(Fl_t^{\xi}(x)) \cdot f \quad \Leftarrow$

$\cdot\ \alpha(0,s) = (TFl_s^{\xi}\,\eta(x)) \cdot f = \eta(x) \cdot (f \circ Fl_s^{\xi}) \quad \Leftarrow$

$$\frac{\partial}{\partial t}\alpha(0,0) = \frac{d}{dt}\Big|_{t=0} \eta\left(Fl_t^{\xi}(x)\right).f = \frac{d}{dt}\Big|_{t=0} (\eta.f)\left(Fl_t^{\xi}(x)\right)$$

$$= \xi_x.(\eta.f).$$

$$\frac{\partial}{\partial s}\alpha(0,0) = \eta(x).\frac{d}{ds}\Big|_{s=0}\left(f\circ Fl_s^{\xi}\right) = \eta_x.(\xi.f).$$

$$\Longrightarrow \frac{d}{dt}\Big|_{t=0}\alpha(t,-t) = \xi_x.(\eta.f) - \eta_x.(\xi.f) = \left(\xi.(\eta.f)-\eta.(\xi.f)\right)_x$$

$$= \underline{[\xi,\eta](x)}.f$$

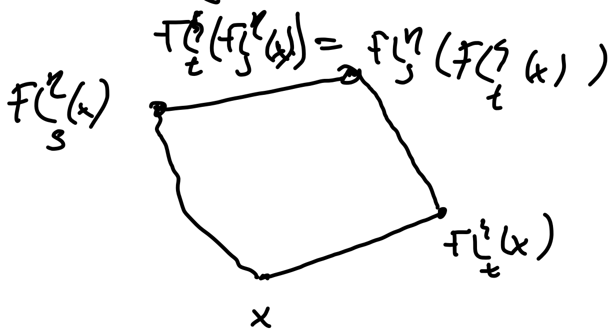
$$= \underline{\left(\frac{d}{dt}\Big|_{t=0} TFl_{-t}^{\xi}\circ\eta\circ Fl_t^{\xi}\right)(x)}.f.$$

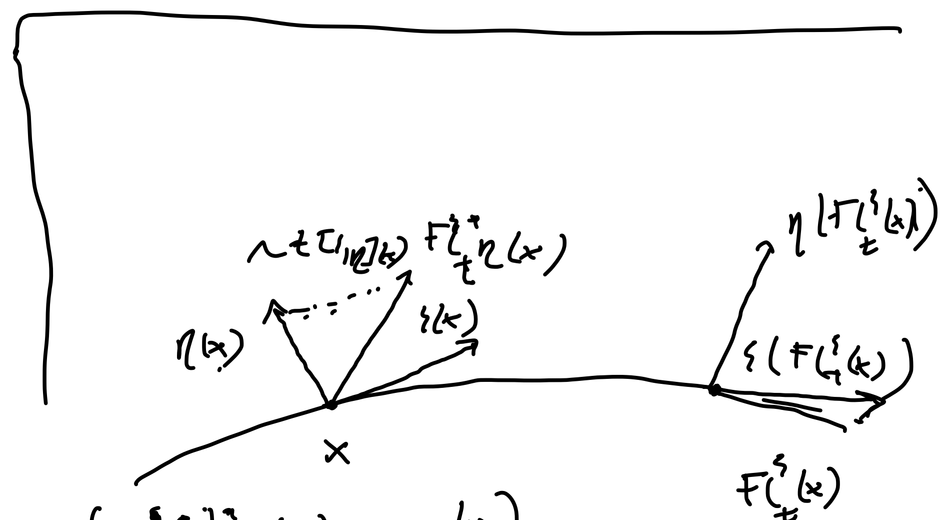
$\square$

Cor. 3.31 $\quad \xi, \eta \in \mathfrak{X}(M)$

$[\xi, \eta] = 0 \iff Fl_t^{\xi *} \eta = \eta$ whenever defined $\iff$

$Fl_t^{\xi} \circ Fl_s^{\eta} = Fl_s^{\eta} \circ Fl_t^{\xi}$ whenever defined.





Proof. see Tutorial.

$$\lim_{t \to 0} \frac{(Fl_t^{\xi *})\eta(x) - \eta(x)}{t} = [\xi, \eta](x)$$

Def. 3.32 M, N mfds., $f : M \to N$ C^∞-map.

Then $\xi \in \mathfrak{X}(M)$ and $\eta \in \mathfrak{X}(N)$ are f-related, if

$$T_x f \, \xi(x) = \eta(f(x)) \quad \forall x \in M.$$

Remark Given a vector field $\eta \in \mathfrak{X}(N)$ (or $\xi \in \mathfrak{X}(M)$) there is in general no vector field $\xi \in \mathfrak{X}(M)$ (resp. $\eta \in \mathfrak{X}(N)$) so that they are f-related. If f is a local diffeom and $\eta \in \mathfrak{X}(N)$, then $\exists!$ f-related vf, namely $f^*\eta$.

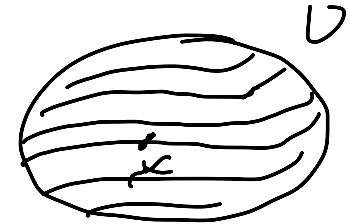
Prop. 3.33 - $f: M \to N$ C^∞-map between mfds.

Suppose $\xi_1, \xi_2 \in \mathfrak{X}(M)$ are f-related to $\eta_1 \in \mathfrak{X}(N)$ resp. $\eta_2 \in \mathfrak{X}(N)$.

Then $[\xi_1, \xi_2]$ is f-related to $[\eta_1 \eta_2]$.

Proof see Tutorial.

3.7 Frobenius Theorem



Existence of flows of vector fields revisited:

$\xi \in \mathfrak{X}(M)$

- for $x \in M$ $\exists$ an integral curve $c: I \to M$, $0 \in I$, $c(0) = x$ ($c(t) = Fl^{\xi}_t(x)$.)

- If $\xi(x) = 0$, then $c(t) = x$ constant curve.
- If $\xi(x) \neq 0$, then $\xi(y) \neq 0$ $\forall y \in U$, U neighbh. of x.

$\Rightarrow$ integral curve through x is a 1-dim. submf of M

Hence, ξ decomposes U into a union of 1-dim. submfds given by the images of the integral curves through $y \in U$.

The tangent space of such a submfd through $y \in U$ equals $\mathbb{R}\xi(y) \subseteq T_y M$.

- If we replace ξ by $f\xi$ for a nowhere vanishing $f \in C^\infty(M, \mathbb{R})$ then the integral curves of $f\xi$ and ξ are just reparametrizations of each other; hence the define the same family of 1-dim. submfds. (of U).

- Suppose $\Delta : x \longmapsto \ell_x \subseteq T_x M$ is a map that assigns to each $x \in M$ a line ℓ_x through (i.e. a 1-dim. subspace of $T_x M$) s.t. $\exists$ an open cover $\{U_i\}$ of M and local vector fields $\xi_i \in \mathfrak{X}(U_i)$ s.t. $\xi_i(y)$ spans ℓ_y $\forall y \in U_i$ $\forall i$.

Then for each $x \in M$ $\exists!$ local smooth submfd. $N_x \subseteq M$ s.t.

$$T_y N_x = \mathcal{E}_y \subseteq T_y M \quad \forall y \in N_x .$$

<u>Def 3.34</u>. M mfd of dim. n.

① A distribution E of rank k on M is given by a k-dim. subspace $E_x \subseteq T_x M$ for each $x \in M$.

② A (smooth) <u>section of $E \subseteq TM$</u> is a vector field s of M s.t. $s(x) \in E_x \quad \forall x \in M$. A local section of E defined on open subset $U \subseteq M$ is a local vector field $s \in \mathfrak{X}(U)$ s.t. $s(x) \in E_x \quad \forall x \in U$.

③ A distribution $E \subseteq TM$ of rank k is called smooth, if for any $x \in M$ $\exists$ an open neighbhd. U of x and local sections $s_1, \dots, s_k \in \mathfrak{X}(U)$ of E s.t. $\{s_1(y), \dots, s_k(y)\}$ is a basis for E_y $\forall y \in U$. Such collection of local sections is called a local frame of E.

A smooth distribution is also called a (smooth) vector subbundle of TM.

④ A distribution $E \subseteq TM$ is called involutive, if for any local sections s, η of E their Lie bracket $[s, \eta]$ is also a local section of E.

(5) A distribution $E \subseteq TM$ is called integrable, if for each $x \in M$ $\exists$ a smooth submfd $N \subseteq M$ with $x \in N$ s.t. for any $y \in N$

$$T_y N = E_y \subseteq T_y M .$$

Such submfds are called integral submfds of E.

Existence of flows for vector fields implies

<u>Prop. 3.35</u> Any smooth distribution of rank 1 on a mfd is integrable.

Distributions of higher rank are not always integrable.

A necessary condition for integrability of a distribution is involutivity:

Let $E \subseteq TM$ be a integrable distribution and $N \subseteq M$ is an integral submfd, i.e. $T_x N = E_x \subseteq T_x M \quad \forall x \in N$.

Assume ξ, η are local sections of E defined on open neighbh. U of $x \in N$ in M. Replacing N by $N \cap U$, we may assume $N \subset U$.

$\xi|_U$ and $\eta|_U$ are i-related to vector fields $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(N)$)

where $i: N \hookrightarrow U \subseteq M$ is the inclusion. $(T_y i : T_y N = E_y \hookrightarrow T_y M$ standard incl.$)$

$\Longrightarrow$ $[\xi|_U, \eta|_U]$ is i-related to $[\tilde{\xi}, \tilde{\eta}] \in \mathfrak{X}(N)$.

Prop. 3.34

$\Longrightarrow$ $\underline{[\xi, \eta](y)} \in \operatorname{im}(T_y i) = \underline{E_y}$ $\forall y \in N$.

Frobenius Thm. shows that also the converse is true, i.e. any involutive smooth distribution is integrable.