(M, g) is a Riem. mfd.

$\rightsquigarrow$ $\exists!$ torsion-free affine connection ∇ s.t. $\underline{\underline{\nabla g = 0}}$

Levi-Civita connection of (M, g).

- $\nabla g = 0 \Longrightarrow$ induced parallel transport is orthogonal.
- Formula for the Christoffel symbols in terms of g.
  (torsion free $\Longleftrightarrow$ $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$).

Prop. 6.29 (M,g) Riem. mfd with Levi-Civita conn. ∇.

Then the Riemannian curv. $R \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(TM))$.

① As $\binom{0}{4}$-tensor (via the metric) it is a section

of $\underline{\Lambda^2 T^*M \otimes \Lambda^2 T^*M} \subseteq \Lambda^2 T^*M \otimes T^*M \otimes T^*M$:

$$g(R(\xi,\eta)(\underbrace{\varrho), \vartheta}_{}) \Leftarrow \quad \forall\, \xi,\eta,\varrho,\vartheta \in \Gamma(TM)$$

$$\shortparallel$$

$$-\,g(R(\xi,\eta)(\vartheta), \varrho). \Leftarrow$$

② $$R(\xi,\eta)(\varrho)+R(\eta,\varrho)(\xi)+R(\varrho,\xi)(\eta)=0$$

„Bianchi identity" $\forall \xi,\eta,\tau \in \Gamma(TM)$

(it holds for any torsion-free affine connection).

③ $$g(R(\xi,\eta)(\varrho),\vartheta)=g(R(\varrho,\vartheta)(\xi),\eta) \quad \text{(Pair symmetry)}$$

(i.e. $g(R(\_,\_)(\_),\_) \in \Gamma(\underline{S^2\Lambda^2T^*M})$)

Proof.

$\Updownarrow \nabla g = 0$

① Follows from (*) of Thm. 6.25.

$$g(\nabla_\xi \nabla_\eta e, v) = -g(\nabla_\eta e, \nabla_\xi v) + \xi \cdot g(\nabla_\eta e, v)$$

$$= g(e, \nabla_\eta \nabla_\xi v) - \eta \cdot g(e, \nabla_\xi v) - \xi \cdot g(e, \nabla_\eta v) + \eta \cdot \xi \cdot g(e, v).$$

$$\Longrightarrow g(R(\xi,\eta)(e), v) = -g(R(\xi,\eta)(v), e) + \cancel{\eta \cdot \xi \cdot g(e,v)} - \cancel{\xi \cdot \eta \cdot g(e,v)} + \cancel{[\xi,\eta] \cdot g(e,v)}$$

(2) Holds for any torsion-free affine connection

Exercise: $[\xi,\eta] = \nabla_\xi \eta - \nabla_\eta \xi$.

(3) Follows from (1) and (2) $=$ $R_g(\xi,\eta,\varrho,v) = g(R(\xi,\eta)(\varrho),v)$.

- $R_g(\xi,\cancel{\eta,\varrho},v) + R_g(\eta,\cancel{\varrho,\xi},v) + R(\varrho,\xi,\eta,v) = 0$
- $R_g(\eta,\cancel{\varrho,v},\xi) + R_g(\varrho,\cancel{v,\eta},\xi) + R_g(v,\eta,\varrho,\xi) = 0$
- $R_g(\varrho,\cancel{v,\xi},\eta) + R_g(v,\cancel{\xi,\varrho},\eta) + R_g(\xi,\varrho,v,\eta) = 0$
- $R_g(v,\cancel{\xi,\eta},\varrho) + R_g(\xi,\cancel{\eta,v},\varrho) + R_g(\eta,v,\xi,\varrho) = 0$

Adding them gives:

$$2R_g(\underline{\varrho},\varsigma,\underline{\eta},\vartheta) = 2R_g(\underline{\eta,\vartheta,\varrho,\varsigma})$$

$\square$.

<u>Prop. 6.30</u> Suppose (M, g^M) and (N, g^N) are Riem. mfds with Levi-Civita connections ∇^M and ∇^N and let $f : (M, g^M) \to (N, g^N)$ be isometry.

(1) $f^*(\nabla^N_\varsigma \eta) = \nabla^M_{f^*\varsigma} f^*\eta \qquad \forall \varsigma, \eta \in \Gamma(TN)$.

(2) $f^* R^N = R^M$ (equiv. $(f^{-1})^* R^M = R^N$).

(3) f map geodesics to geodesics and $f \circ \exp^M_x = \exp^N_{f(x)} \circ T_x f$. (where defined).

Proof.

① see Thm. 6.14 (where we proved this for hypersurfaces).

② Follows from ① and def. of R^M and R^N

$$\left(R^M(f^*\xi, f^*\eta)(f^*e) = f^*(R^N(\xi,\eta)(e)) \quad \forall \xi,\eta,e \in \Gamma(TN) \right).$$

③ For a curve $c: I \to M$, $f\circ c$ is a curve in N and for a vector field η along c, $t \mapsto T_{c(t)}f\, \eta(t)$ is a vector field along $f\circ c$.

① $\Longrightarrow$ $T_{c(t)}f(\nabla^M_{c'}\eta)(t) = \nabla^N_{f\circ c}\, T_{c(t)}f\, \eta(t)$.

In particular, $(T_{c(t)}f\, \nabla^M_{c'}c')(t) = (\nabla^N_{(f\circ c)'}(f\circ c)')(t) \Longrightarrow$

f maps geodesics to geodesics.

$c: t \mapsto \exp^M_{p_x}(t\xi_x)$ is a geodesic with $c(0) = x$ and $c'(0) = \xi_x$

and hence $\tilde{c}: t \mapsto (f \circ \exp^M_{p_x})(t\xi_x)$ is a geodesic with $\tilde{c}(0) = f(x)$

and $\tilde{c}'(0) = T_x f \, \xi_x$. By uniqueness, $\tilde{c}(t) = \exp^N_{f(x)}(T_x f \, t\xi_x)$

$\square$.

Cor. 6.31 Suppose (M,g) is a connected Riem. mfld. and $f: (M,g) \to (M,g)$ an isometry. If $\exists\, x \in M$ such that $f(x) = x$ and $T_x f = \mathrm{Id}_{T_xM}$, then $f = \mathrm{Id}_M$.

Def. 6.32 (M, g) is a Riem. mfd. A vector field ξ on M is called a Killing field (or infinitesimal isometry), if

$$\mathcal{L}_\xi g = 0$$

$$\frac{d}{dt}\Big|_{t=0} (Fl_t^\xi)^* g$$

$$\left(\iff Fl_t^{\xi\,*} g = g \text{ wherever defined} \right.$$

$$\left. \text{i.e. } Fl_t^\xi \text{ is a local isometry} \right).$$

Prop. 6.33 (M,g) Riem. mfd., ∇ Levi-Civita connection.

① A vector field $\xi \in \Gamma(TM)$ is Killing

$\Longleftrightarrow$

$$g(\nabla_\eta \xi, e) + g(\nabla_e \xi, \eta) = 0$$ (Killing equation)

(i.e. $\binom{0}{2}$-tensor $g(\nabla_{\_}\xi, \_)$ is skew-symmetric).

② The set of Killing vector fields is a finite-dim. subalgebra of $(\Gamma(TM), [\,,\,])$.

Proof.

① $$(\mathcal{L}_\xi g)(\eta, e) = \xi\cdot g(\eta, e) - g(\underbrace{\mathcal{L}_\xi \eta}_{=[\xi,\eta]}, e) - g(\eta, \underbrace{\mathcal{L}_\xi e}_{=[\xi,e]})$$

$$\underset{\text{torsion-freeness of }\nabla}{=} \xi\cdot g(\eta, e) - g(\nabla_\xi \eta, e) + g(\nabla_\eta \xi, e) - g(\eta, \nabla_\xi e) + g(\eta, \nabla_e \xi)$$

$$\underset{\nabla g = 0}{=} g(\nabla_\eta \xi, e) + g(\eta, \nabla_e \xi)$$

(∗), Thm. 6.25

② Killing equation is linear overdetermined system of PDEs.

$\Longrightarrow$ solution space is a subspace of $\Gamma(TM)$.

ξ, η are Killing fields

$$\mathcal{L}_{[\xi,\eta]} g = \underbrace{\mathcal{L}_\xi \mathcal{L}_\eta g}_{=0} - \underbrace{\mathcal{L}_\eta \mathcal{L}_\xi g}_{=0} = 0$$

$\Rightarrow$ set of Killing fields is subalgebra of $(\Gamma(TM), [\cdot,\cdot])$.

Finit-dimensionality follows from Cor. 6.30:

$\xi(x)$, $(\nabla \xi)(x)$ determines ξ on any connected component.

$$n + \underbrace{\frac{n(n-1)}{2}}_{\dim(o(n))} = \frac{n(n+1)}{2} \qquad \dim(M) = n$$

Finite-dimensionality of algebra of Killing fields is key to:

<u>Thm. 6.34</u> (M, g) is a Riem. mfd. of dim. n.

Then the isometry group $\mathrm{Isom}(M,g) = \{ f : (M,g) \to (M,g) : f \text{ isometry} \}$

(group w.r. to composition of maps.) is in a natural way a Lie group of dim. $\leq \frac{n(n+1)}{2}$.

Its Lie algebra is $T_{id}\,\mathrm{Isom}(M,g) = \{ \xi \in \Gamma(TM) : \xi \text{ is Killing and complete} \}$

with the Lie bracket given by the negative of the Lie bracket of vector fields.

Ex. 1 $(\mathbb{R}^n, g_{euc})$

We have seen that $F(x) = Ax + b$ $\quad A \in O(n)$, $b \in \mathbb{R}^n$

is an isometry of $(\mathbb{R}^n, g_{euc})$.

$$\mathrm{Euc}(n) = \{F : \mathbb{R}^n \to \mathbb{R}^n : F(x) = Ax + b \text{ for } A \in O(n), b \in \mathbb{R}^n\}$$
$$= \mathrm{Isom}(\mathbb{R}^n, g_{euc}) \quad \text{by Cor. 6.31.}$$

$F : \mathbb{R}^n \to \mathbb{R}^n$ isometry, $F(0) = b \in \mathbb{R}^n$

$\Longrightarrow$ $\tilde{F} = F - b$ is an isometry with $\tilde{F}(0) = 0$.

$T_0 \tilde{F} =: A \in O(n)$

$\Rightarrow$ $\tilde{F} \circ A^{-1}$ is an isometry as a composition of isometries

and $(\tilde{F} \circ A^{-1})(0) = 0$

$$\Rightarrow T_0(\tilde{F} \circ A^{-1}) = T_0\tilde{F} \circ T_0 A^{-1} = A \circ A^{-1} = \mathrm{Id}_{\mathbb{R}^n}$$

$$\Rightarrow \underbrace{\tilde{F} \circ A^{-1}}_{(F-b)} = \mathrm{Id}_{\mathbb{R}^n} \Leftrightarrow F = A + b$$

$$\dim(\mathrm{Euc}(n)) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Ex.2 (S^n, g_{rd})

By construction of g_{rd} (being induced by $\langle , \rangle$ on $\mathbb{R}^{n+1}$)

$O(n+1) \subseteq \mathrm{Isom}(S^n, g_{rd})$.

In fact, $O(n+1) = \mathrm{Isom}(S^n, g_{rd})$ again by Cor. 6.3 1

$e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$, $F \in \mathrm{Isom}(S^n, g_{rd})$.

$\exists\, A \in O(n+1)$ s.t. $A F(e_1) = e_1$.

$A \circ F$ is an isometry fixing e_1, $\underbrace{T_{e_1}(A \circ F)}_{\exists!\, B \in O(n)} = T_{e_1}S^n \longrightarrow T_{e_1}S^n$

$T_{e_1}S^n = e_1^{\perp} \cong \mathbb{R}^n$

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & B \end{array}\right) \in O(n+1)$$

$$\tilde{F} := \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & B \end{array}\right) \circ A \circ F$$ is an isometry fixing e_1

and $T_{e_1}\tilde{F} = \mathrm{Id}_{T_{e_1}S^n}$.

$$\Longrightarrow \tilde{F} = \mathrm{Id}_{S^n} \Leftrightarrow F \in A^{-1} \circ \left(\begin{array}{c|c} 1 & \\ \hline & B \end{array}\right) \in O(n+1).$$

Ex. 3 Hyperbolic space.

$\mathbb{R}^{n+1}$ $(x^0, x^1, \dots, x^n)$ coordinates.

equipped with the Lorentzian inner product (of sign. $(1,n)$)

$$\langle x,y\rangle := -x^0y^0 + \sum_{i=1}^{n} x^iy^i = x^t \begin{pmatrix}-1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix} y$$

(Minkowski space).

$$H^n = \{x \in \mathbb{R}^{n+1} : \underline{\langle x,y\rangle = -1}, \ \underline{x^0 > 0}\} \subseteq \mathbb{R}^{n+1}$$

is an n-dim. submfd. $T_xH^n = x^\perp$ $(g_{hyp})_x$

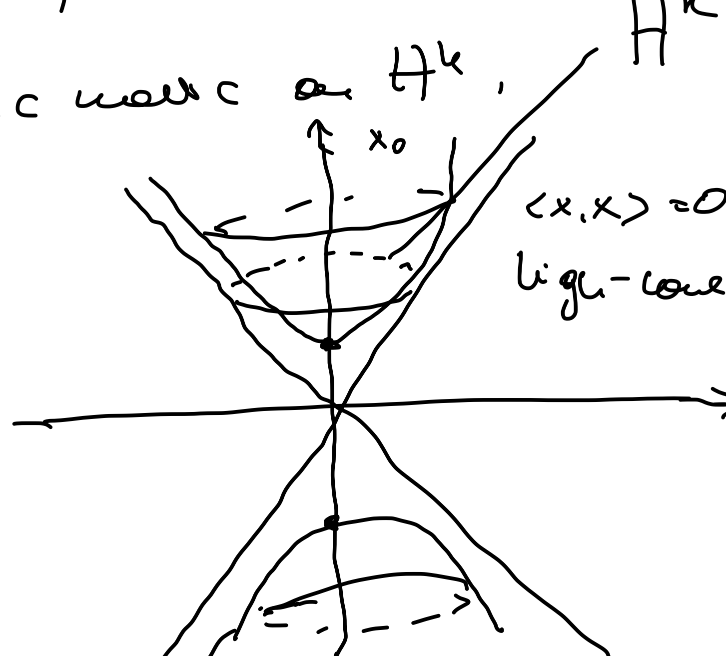
Lorentzian inner product induces positive definite inner product on

$T_x H^n = x^\perp \quad \forall x \in H^n$

$\quad\quad\;\; \parallel$

$\{ y \in \mathbb{R}^{n+1} : \langle x, y \rangle = 0 \}$ ← Lorentzian,

$\leadsto$ g_{hyp} is Riem. metric on H^n, called the standard hyperbolic metric on H^n,



What are geodesics?

What is the isometry group?

Proof. We know that $\exp_x : U \to \tilde{U}$ is a diffeom. for sufficiently small neighb. U of $0 \in T_xM$ and $\tilde{U}$ of $x \in M$.

By (3) of Prop. 6.29, $f \circ \exp_x = \exp_{f(x)} \circ T_x f = \exp_x$

$$\Longrightarrow f\big|_{\exp_x(U) = \tilde{U}} = \mathrm{Id}_{\tilde{U}}$$

$$\Longrightarrow \hat{M} := \{ y \in M : f(y) = y,\ T_y f = \mathrm{Id}_{T_yM} \}$$

is a non-empty (since $x \in \hat{M}$) open subset.

But it is obviously also closed. By connectedness of M, $M = \hat{M}$.