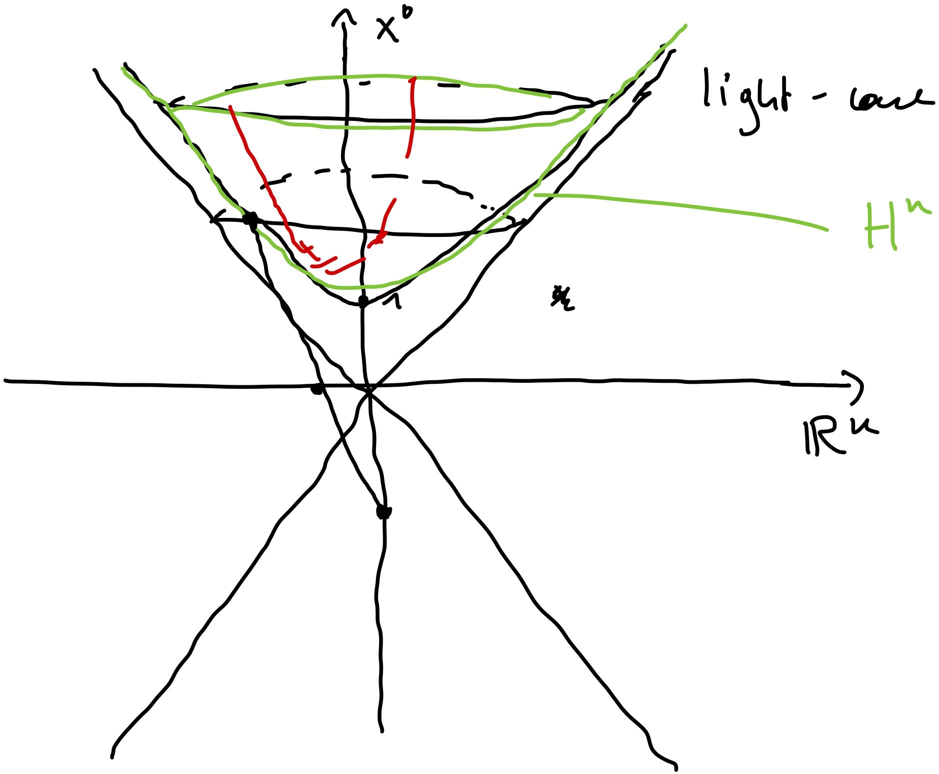
Ex. 3

$$(H^n, g_{hyp}) \subseteq (\mathbb{R}^{n+1}, \langle , \rangle)$$

Lorentzian inner prod.

$$(x^0, x^1, \dots, x^n)$$



Geodesics of (H^n, g_{hyp}) ?

$x \in H^n$, $\xi_x \in T_x H^n = x^\perp$ and let $c: (-\varepsilon, \varepsilon) \to M$ be the unique geod. with $c(0) = x$, $c'(0) = \xi_x \neq 0$

Let P be the 2-dim. subspace spanned by x and $\frac{\xi_x}{\|\xi_x\|}$

$$P = \left\{ tx + \frac{r}{\|\xi_x\|}\xi_x \;,\; t, r \in \mathbb{R} \right\}$$

$\mathbb{R}^{n+1} = \underline{P} \oplus \underline{P^\perp}$ ($P^\perp$ orthog complem. w.r. to $\langle , \rangle$)

$S : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ linear map $S(x) = \begin{cases} x & \text{for } x \in P \\ -x & \text{for } x \in P^\perp \end{cases}$

$s \in O(1, n)$, that is, it is a linear isometry of Minkowski space $(\mathbb{R}^{n+1}, \langle , \rangle)$

Moreover, s leaves H^n invariant, hence s restricts to an isometry of (H^n, g_{hyp}).

$s \circ c$ is also geod. of H^n by Prop. 6.30.

Since $(s \circ c)(0) = x$ and $(s \circ c)'(0) = T_x s \, c'(0) = s \xi_x = \xi_x$

By uniqueness, $s \circ c = c$.

$\Rightarrow$ c equals the intersection of P with H^n.

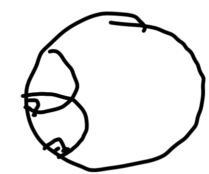
$$\Longrightarrow \quad c(t) = \cosh(\|\xi_x\| t) x + \sinh(\|\xi_x\| t) \frac{\xi_x}{\|\xi_x\|}.$$

Isometry group of (H^n, g_{hyp}) equals

$$O_+(1,n) = \{A \in O(1,n) : A(H^n) = H^n\}.$$

($A \in O(1,n)$: A either $A(H^n) = H^n$ or $A(H^n) = -H^n$).

<u>Remark</u> Other models of (H^n, g_{hyp})


geodesics

· Poincaré ball model : $H^n \xrightarrow{\sim} B^n$ projection to subspace $x^0 = 0$

$$B^n = \{x \in \mathbb{R}^n : \|x\|_{\text{euclid.}} < 1\}$$

project a point $x \in H^n$ to intersection of $x^0 = 0$ with the line between x and $(-1, 0 \ldots 0)$.

- Poincaré-half space model.

Since equation for Killing vector fields is an overdetermined system of PDEs, on a general Riemannian mf $\nexists$ Killing fields (isometry group is trivial).

Hence, Riem. mf. with large isometry group need to be very special.

Thm. 6.35 Suppose (M, g) is simply-connected Riem. mfd. of dim. n such that $\dim(\mathrm{Isom}(M,g)) = \frac{n(n+1)}{2}$.

Then (M, g) is isometric (up to rescaling of the metric by a positive constant) to either of these three:

$$\left.\begin{array}{ll} \bullet\ (\mathbb{R}^n, g_{euc}) & \text{Euclidean space} \\ \bullet\ (S^n, g_{rd}) & \text{sphere.} \\ \bullet\ (H^n, g_{hyp}) & \text{hyperbolic space.} \end{array}\right\} \quad (*)$$

Remark

- Isometry groups of (*) act transitively on (M,g):
  for any $x,y \in (M,g)$ $\exists f \in \mathrm{Isom}(M,g)$ s.t. $f(x)=y$.

$\to$ they are homogeneous Riem. mfd:

$$(\mathbb{R}^n, g_{euc}) \simeq \mathrm{Euc}(n) / O(n)$$

$$(S^n, g_{rd}) \simeq O(n+1) / O(n)$$

$$(\mathbb{H}^n, g_{hyp}) \simeq O_+(1,n) / O(n).$$

- They are the unique (up to constant positive rescaling of g)

the ~~the~~ unique complete simply-connec. Riem. mfd. of constant sectional curvature $(0, 1, -1)$.

- All three metrics are Einstein and locally conformally equiv. to each other.

<u>Significance of Riemann curvature</u>:

Note that $R_x = 0 \ \forall x \in (\mathbb{R}^n, g_{euc})$. Conversely, one has:

Thm. 6.36 Suppose (M,g) is a Riem. mfd. and $x \in M$. Then the following statements are equivalent:

① The Riemannian curv. R vanishes on an open neighbhd. of x.

② $\exists$ a chart (U,u) around x s.t. the coordinate vector fields $\{\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}\}$ form an orthonormal basis for T_yM, $\forall y \in U$.

③ $\exists$ an open neigh. U of x and an isometry $f: (U,g) \to (V, g_{euc})$, where $V \subseteq \mathbb{R}^n$ open subset.

Proof Evidently, ② ⟺ ③ : If ②, $u : U \to u(U)$ is an isometry, and if ③ $f : (U, g) \xrightarrow{\sim} (V, g_{euc})$ is a chart with the required property.

Since curv. of $(\mathbb{R}^n, g_{euc})$ vanishes and ② of Prop. 6.30 holds, also ③ ⟹ ①.

It remains to show that ① ⟹ ② :

Since this is a local question it suffices to prove it locally around $0 \in \mathbb{R}^n$ for an arbitrary metric g on $\mathbb{R}^n$ with vanishing curvature.

We write $(x^1, \dots, x^n)$ for the coordinates in $\mathbb{R}^n$.

Choose an orthonormal basis $\{\xi_1(0), \dots, \xi_n(0)\}$ of $T_0\mathbb{R}^n$ w.r.t. to g.

We can extend them to local vector fields on $\mathbb{R}^n$ as follows:

Fix i: To get $\xi_i(x^1, \dots, x^n)$ first parallely transport $\xi_i(0)$ along the line $t \longmapsto (t, 0 \dots 0)$ to the point $(x^1, 0 \dots, 0)$, then parallely transport $\xi_i(x^1, 0 \dots, 0)$ along the line $t \to (x^1, t, 0 \dots 0)$ to $(x^1, x^2, 0 \dots 0)$ and so on.

Claim. $\zeta_1, \dots, \zeta_n$ are parallel w.r.t. the Levi-Civ. conn. ∇.

By construction, ζ_i is parallel along all lines $t \mapsto (y^1, \dots, y^{n-1}, t)$

$\Rightarrow \underline{\nabla_{\frac{\partial}{\partial x^n}} \zeta_i = 0}$. Same argument shows that $\nabla_{\frac{\partial}{\partial x^{n-1}}} \zeta_i$ vanishes on the subspace of points $\underline{(y^1, \dots, y^{n-1}, 0)}$.

Now $\left[\frac{\partial}{\partial x^{n-1}}, \frac{\partial}{\partial x^n}\right] = 0$ and $R\left(\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^{n-1}}\right) = 0$

imply $\underline{\underline{\nabla_{\frac{\partial}{\partial x^n}} \nabla_{\frac{\partial}{\partial x^{n-1}}} \zeta_i}} = \nabla_{\frac{\partial}{\partial x^{n-1}}} \underline{\underline{\nabla_{\frac{\partial}{\partial x^n}} \zeta_i}} = 0$

$\Longrightarrow \nabla_{\frac{\partial}{\partial x^{n-1}}} \xi_i$ is parallel along all lines $t \mapsto (y^1, \dots, y^{n-1}, t)$

and vanishes at $t = 0$. Hence, $\nabla_{\frac{\partial}{\partial x^{n-1}}} \xi_i = 0$.

Next, $\nabla_{\frac{\partial}{\partial x^{n-2}}} \xi_i$ vanishes at all points $\underline{(y^1, \dots, y^{n-2}, 0, 0)}$

and vanishing of R implies (as above) that it vanishes at all points.

Iteratively, one gets $\nabla_{\frac{\partial}{\partial x^j}} \xi_i = 0 \quad \forall\, i, j$.

In particular, $\nabla_{\xi_j} \xi_i = 0 \quad \forall i,j \implies [\xi_i, \xi_j] = 0 \quad \forall i,j.$

torsion-freeness of ∇

Hence, Lemma 3.30 (before Frob. Thm.), $\exists$ a chart (U, u) around 0 s.t. $\xi_i\big|_U = \frac{\partial}{\partial u^i}$,

which implies (2) by Prop. 6.28.

$\square$.

Def. 6.37 (M,g) Riem. mfd., $c : [a,b] \to M$ C^∞-curve.

Then the (arc) length of c (w.r.t. to g) is given by

$$L(c) := \int_a^b \|c'(t)\|_g \, dt = \int_a^b \sqrt{g(c'(t), c'(t))} \, dt$$

(For a piecewise C^∞-curve (i.e. $a = t_0 < \dots < t_n = b$ s.t. $c|_{[t_i, t_{i+1}]} =: c_i$ is C^∞ $\forall i$, $L(c) = \sum_i L(c_i)$.)

For a geodesic c, $\|c'(t)\|_g$ = constant in t (Prop. 6.28)

$\Rightarrow$ Geodesics are parametrized proportional to arc length:

$$L(c) = \int_0^a \|c'(t)\|\,dt = \text{const}\,(b-a).$$

Recall also that if $\phi : [a',b'] \to [a,b]$ a diffeo. with $\phi'(t) > 0$, then $L(c) = L(c\circ\phi)$.

<u>Thm. 6.38</u> (M,g) a connected Riem. mfd. For $x,y \in M$ let

$$d_g(x,y) := \inf\{L(c) : c : [0,1] \to M \text{ piecewise smooth with } c(0)=x \text{ and } y = c(1)\}.$$

Then (M, d_g) is a metric space and the induced topology coincides with the mfd. topology.

Proof. see literature.

Minimizing properties of geodesics:

Def. 6.38 (M, g) Riem. mfd. We call a curve $c : [a,b] \to M$ minimizing, if it is the shortest path between its endpoints, i.e. $d_g(c(a), c(b)) = L(c)$.

Def. 6.39 (M,g) Riem. mfd., $x \in M$.

① A normal neigh. U of x is of the form

$U = \exp_x(V)$ for an open neigh. of $0 \in T_xM$.

② By Thm. 6.23, $\exists\, \varepsilon > 0$ s.t. $\exp_x : B_\varepsilon(0) \to U$

is a diffeo. from $B_\varepsilon(0) = \{ \xi_x \in T_xM : \|\xi_x\|_g < \varepsilon \}$

to an open neigh. U of x. $U = \exp_x(B_\varepsilon(0)) =: B_\varepsilon(x)$

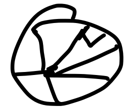
is called geodesic (or normal) ball with center

x and radius ε.

$y \in B_\varepsilon(x)$ can be joined to x by geod. $c : [0,1] \to M$

$\shortparallel$ $\exp_x(\xi_x)$. $\qquad c(t) = \exp_x(t\xi_x)$

Geodesics in $B_\varepsilon(x)$ emanating from x are called radial geodesics.



(3) For a $0<\delta<\varepsilon$ with ε as in (2),

$$S_\delta(x) = \exp_x\left(S_\delta(0)\right) \quad \text{with } S_\delta(0) = \{\xi_x \in T_xM : \|\xi_x\| = \delta\}$$

is called the geodesic (or normal) sphere at x of radius δ.

Lemma 6.40 (Gauß Lemma) Let $x \in M$ and $v \in T_x M$ s.t. $\exp_x(v)$ is defined. Let $w \in T_x M$.

Then

$$g\left(T_v \exp_x v,\ T_v \exp_x w\right) = g(v, w) \quad (*)$$

where $T_v \exp_x : T_v T_x M \to T_x M$ is viewed as a map $T_x M \to T_x M$ via $T_v T_x M \cong T_x M$.

Proof $v \in T_x M$ $\qquad T_x M = \mathbb{R} v \oplus v^{\perp}$ (w.r.t. g.)

$$w = w_t \oplus w_n \quad \text{w.r.t. this decomp.}$$

Since $T_v \exp_x$ is linear and by def. of $\exp_x$,

$$g\left(T_v \exp_x v, T_v \exp_x w_t\right) = g(v, w_t)$$

$$c'(1) = T_x \exp_x v$$

$$c'(0) = v$$

($\cancel{g}$ ($c : t \mapsto \exp_x(tv)$)

$$g(c'(0), c'(0))$$

$$= g(c'(0), c'(0)) = g(v, v).$$

It suffices to prove (*) for $w = w_\perp \neq 0$.

Since $\exp_x(v)$ is ord. $\exists\, \varepsilon > 0$ s.t. $\exp_x(u)$ is ord. for $u = t\,v(s)$ $\quad 0 \leq t \leq 1, \; -\varepsilon < s < \varepsilon$.

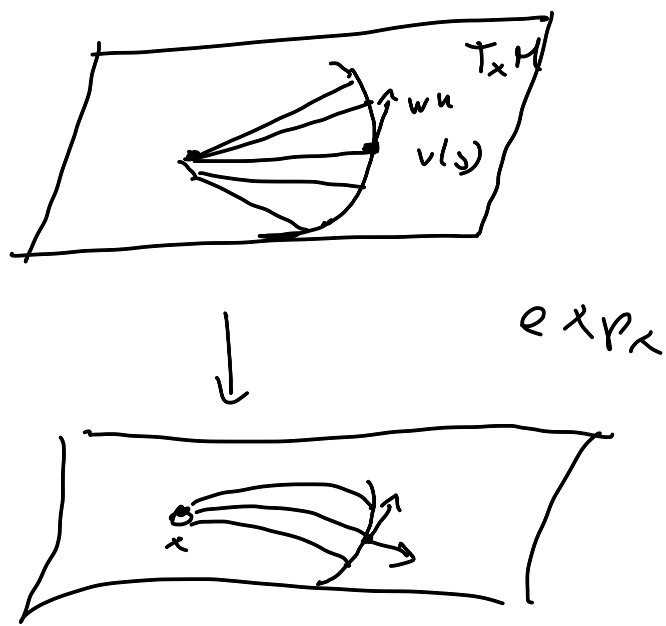
where $v(s)$ is a curve in T_xM with $v(0) = v$, $v'(0) = w_\perp$ and $\|v(s)\|_g = $ constant.

Hence, we can consider the param. surface

$$f : A \to M \qquad A = [0,1] \times (-\varepsilon, \varepsilon)$$

$$\cancel{f} \quad f(t,s) = \exp_x(t\,v(s)).$$

$t \longmapsto f(t, s_0)$ are geodesics for fixed s_0



$$g\left(\frac{\partial f}{\partial s}(1,0), \frac{\partial f}{\partial t}(1,0)\right) = g\left(T_v \exp_{p_x} w_u, T_v \exp_{p_x} v\right)$$

$$\underline{\frac{\partial}{\partial t} \underline{g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)}} \underset{\nabla g = 0}{=} g\left(\nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) + g\left(\frac{\partial f}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial t}\right)$$

$$= 0$$

$$\underset{\substack{\nabla \\ \text{torsionfree}}}{=} g\left(\nabla_{\frac{\partial}{\partial s}} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right)$$

$\frac{\partial f}{\partial t}$ is tangent vector of geodesic.

$$\underset{\nabla g = 0}{=} \frac{1}{2} \frac{\partial}{\partial s} g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) = 0$$

$\|v(s)\|_g = \text{const.}$

Geodesic are locally minimizing:

Prop. 6.41 (M, g) Riem. mfd., $x \in M$, U normal neighb. of x and $B \subset U$ geod. ball with center x. Let $\gamma : [0,1] \to B$ be a geod. with $\gamma(0) = x$ and $c : [0,1] \to M$ a piecewise C^∞-curve $c(0) = x$ and $c(1) = \gamma(1)$, then $L(\gamma) \leq L(c)$. If equality holds, then $\gamma([0,1]) = c([0,1])$.

Proof: Use lemma of Gauß.

Prop. is not globally true (cf. sphere).

On the other hand:

<u>Prop. 6.4.1</u> If a piece-wise smooth curve $c:[a,b]\to M$ with param. proport. to arc length $(L(c) = \text{const.}(b-a))$ is minimizing, then c is a geodesic. In particular, c is smooth.

<u>Proof</u>. see literature.

Thm. 6.42 (Hopf-Rinow Thm.) (M,g) is a connected Riem. mfld. Then the following are equiv.:

① (M,g) is complete

② $\exists\, x \in M$ s.t. $\exp_x$ is defined on all of $T_x M$.

③ (M, d_g) is complete as a metric space.

④ Closed and bounded subsets in M are compact.

In addition, if any of these statements are satisfied then are two

(5) For any $x, y \in M$ $\exists$ a geodesic c joining x and y s.t. $L(c) = d(x,y)$.

<u>Proof</u>. See literature.

$\Longrightarrow$ $\underline{g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)}$ is constant in t.

Since $\lim\limits_{t \to 0} \frac{\partial f}{\partial s}(t,0) = 0$

$$\Longrightarrow \quad g\left(\frac{\partial f}{\partial s}(0,1), \frac{\partial f}{\partial t}(0,1)\right) = 0$$

$\square$.