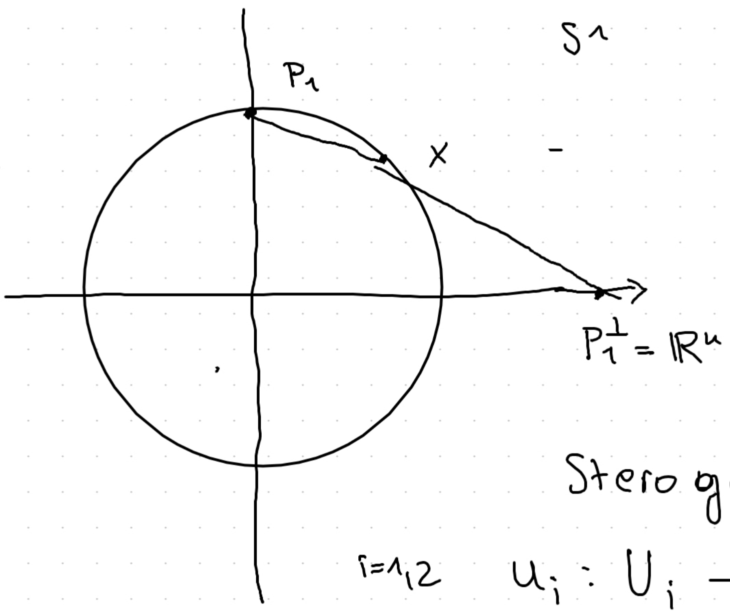
Recall: C^∞- manifold $:\Leftrightarrow$ second countable Hausdorff topolog. space M equipped with a maximal C^∞-atlas of charts $\mathcal{A} = \{(U_i, u_i) : i \in I\}$.

Remark

- Similarly one may define C^k-atlases, $0 \le k \le \infty$ or $k = \omega$ and C^k-manifolds and real analytic manifolds. For $k=0$ one speaks of topological manifold.
- Atlases with values in $\mathbb{C}^n$ and requiring transition maps to be biholomorphisms leads to holom.-atlases and complex manifolds.

Ex By Lemma 2.12, any submanifold of dim. k of $\mathbb{R}^n$ is a k-dim. manifold.

Ex. $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$



Fix $p_1 \in S^n$ as 'north pole' and let $p_2 := -p_1$ the corresp. south pole.

$U_1 := S^n \setminus \{p_1\}$

$U_2 := S^n \setminus \{p_2\}$

Sterographic projection gives us:

$i = 1,2$ $\quad u_i : U_i \longrightarrow \mathbb{R}^n = p_i^\perp$

$$u_i(x) = (1-\lambda)p_i + \lambda x$$

$$= \frac{1}{1-\langle p_i, x\rangle}\left(x - \langle p_i, x\rangle p_i\right)$$

$0 = \langle p_i, p_i + \lambda(x - p_i)\rangle$

$= \underbrace{\|p_i\|^2}_{=1} - \lambda\langle p_i, (x-p_i)\rangle$

$\Rightarrow \lambda = \frac{1}{1-\langle p_i, x\rangle}$

Inverses :

$$u_i^{-1} : \mathbb{R}^n \longrightarrow U_i$$

$$u_i^{-1}(y) = \frac{1}{1+\|y\|^2}\left(2y + (\|y\|^2 - 1)p_i\right)$$

<u>Check</u> : $x = p_i + \mu(y - p_i)$

$$1 = \langle x, x\rangle = \underset{\overset{\|}{1}}{\langle p_i, p_i\rangle} + 2\langle p_i, \mu(y-p_i)\rangle + \mu^2\langle y-p_i, y-p_i\rangle$$

$$\Rightarrow 0 = \mu\left(\mu\,\|y-p_i\|^2 - 2\langle p_i, (p_i - y)\rangle\right)$$

$\mu = 0$ $\quad x = p_i$

$$\mu = \frac{2\langle p_i, (p_i - y)\rangle}{\|y-p_i\|^2} = \frac{2}{\|y\|^2+1}$$

- $S^n = U_1 \cup U_2$
- $u_i : U_i \to \mathbb{R}^n$ is a homeomorphism $i = 1,2$.

Note that $u_1(U_1 \cap U_2) = u_2(U_1 \cap U_2) = \mathbb{R}^n \setminus \{0\}$.

$$u_2 \circ u_1^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$$
$$y \mapsto \frac{y}{\|y\|^2}$$

~~which~~ is smooth.

$\Longrightarrow$ $\mathcal{A} = \{(U_i, u_i) : i = 1,2\}$ is a smooth atlas with values in $\mathbb{R}^n$ of S^n.

Motivated by Thm. 2.14 we define:

<u>Def 2.19</u> M, N smooth manifold with max. atlases $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $f: M \to N$ be a map.

1. f is smooth (or C^∞) at $x \in M$, if $\exists$ charts $(U,u) \in \mathcal{A}$ with $x \in U$ and $(V,v) \in \mathcal{B}$ with $f(x) \in V$ s.t.

$$v \circ f \circ u^{-1}: \underbrace{u(U \cap f^{-1}(V))}_{\subseteq \mathbb{R}^n} \longrightarrow \underbrace{v(V)}_{\subseteq \mathbb{R}^m}$$

is smooth.

Moreover, f is smooth, if f is smooth at all points.

2. f is a diffeomorphism, if f is a smooth bijection with smooth inverse.

We say M and N are diffeomorphic, if $\exists$ a diffeom. between them, <u>Notation</u>: $M \simeq N$.

Remark Note that ① is independent of the choice of charts, since transitions maps are diffeomorphisms:

$$v_2 \circ f \circ u_2^{-1} = \underbrace{(v_2 \circ v_1^{-1})} \circ \underbrace{(v_1 \circ f \circ u_1^{-1})} \circ \underbrace{(u_1 \circ u_2^{-1})}.$$

Def 2.20 Suppose M, N are smooth manifolds with max. atlases $\mathcal{A}$ and $\mathcal{B}$ resp. Let $f: M \to N$ be a map.

① f is a immersion (or submersion) at x, if $\exists$ charts $(U, u) \in \mathcal{A}$ with $x \in U$ and $(V, v) \in \mathcal{B}$ with $f(x) \in V$ s.t. $v \circ f \circ u^{-1}$ is immersion (resp. submersion).

② f is called of constant rank r in an open neighbhd W of $x \in M$, if for every $y \in W$ $\exists$ charts $(U, u) \in \mathcal{A}$, $(V, v) \in \mathcal{B}$ with $y \in U$ and $f(y) \in V$ s.t. the derivative of

$v \circ f \circ u^{-1}$ at $u(y)$ is of rank r.

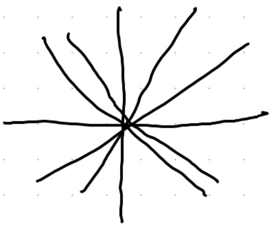
As we have seen any k-dim. submfd. of $\mathbb{R}^n$ is in (a natural way) a k-dim. smooth manifold. In fact, also the converse is true: By a theorem of Whitney, any k-dim. C^∞-manifold is diffeomorphic to a smooth submanifold of $\mathbb{R}^{2k}$.

Ex (Projective space)

$$M := \mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} \big/ \sim \qquad x, y \in \mathbb{R}^{n+1} \setminus \{0\}$$

$$x \sim y \Leftrightarrow \exists \lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \text{ s.t. } x = \lambda y.$$

$$= \mathbb{R}^{n+1} \big/ \mathbb{R}^*$$



= set of lines through zero in $\mathbb{R}^{n+1}$.

n-dim. proj. space.

$\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}P^n$ ↙ homog. coordinates.

$$\pi(x) =: [x] =: [x^1 : \dots : x^{n+1}]$$

- $\mathbb{R}P^n$ is naturally equipped with a topology, namely the quotient top. with resp. to π

  ($U \subseteq \mathbb{R}P^n$ is open $\Leftrightarrow$ $\pi^{-1}(U)$ is open.

  $f : \mathbb{R}P^n \longrightarrow X$ X topolog. space.

  is continuous $\Leftrightarrow$ $f \circ \pi : \mathbb{R}^{n+1} \setminus \{0\} \to X$ is continuous)

- Now set

$$U_i := \{ [x^1 : \dots : x^{n+1}] \in \mathbb{R}P^n : x^i \neq 0 \} \subseteq \mathbb{R}P^n$$

It is open, since $\pi^{-1}(U_i)$ is open in $\mathbb{R}^{n+1}$ and $\mathbb{R}P^n = \bigcup_{i=1}^{n+1} U_i$

- Charts:

$u_i : U_i \longrightarrow \mathbb{R}^n$

$$u_i([x^1 : \dots : x^{n+1}]) = \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\right) \in \mathbb{R}^n$$

It is continuous, since $u_i \circ \pi$ on $\pi^{-1}(U_i)$ is.

Moreover,

$$(x^1, \dots, x^n) \overset{u_i^{-1}}{\longmapsto} [x^1 : \dots : x^{i-1} : 1 : x^i : \dots : x^n]$$

defines a continuous inverse to u_i. $\Rightarrow$ $U_i : U_i \to \mathbb{R}^n$ is a homeom.

- Smoothness of transition maps:

$i \neq j$, $i < j$

$$u_i(U_i \cap U_j) = \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^{j-1} \neq 0\}$$

For $i+1<j$ the transition maps are:

$$u_j \circ u_i^{-1} : u_i(U_i \cap U_j) \to u_j(U_i \cap U_j)$$

$$(x^1, \ldots, x^n) \mapsto \left(\frac{x^1}{x^{j-1}}, \ldots, \frac{x^{i-1}}{x^{j-1}}, \frac{1}{x^{j-1}}, \frac{x^i}{x^{j-1}}, \ldots, \frac{x^{j-2}}{x^{j-1}}, \frac{x^j}{x^{j-1}}, \ldots, \frac{x^n}{x^{j-1}}\right)$$

For $i+1=j$ $\quad (x^1, \ldots, x^n) \mapsto \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{1}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^n}{x^i}\right)$

They are smooth.

Similarly, one verifies that $u_j \circ u_i^{-1}$ are smooth for $i>j$.

$\Rightarrow$ $\mathcal{A} = \{(U_i, u_i) : i = 1, \ldots, n+1\}$ is C^∞-atlas for $\mathbb{R}P^n$.

The category of smooth manifolds respects the operations of taking finite products, disjoints unions and restrictions to open subsets:

Ex. $(M, \mathcal{A})$ manifold.

Then any open subset $U \subseteq M$ is a smooth manifold, since

$$\mathcal{A}|_U := \{ (U_i \cap U, u_i|_{U \cap U_i}) : i \in I \}$$

is an atlas for U.

Ex. $(M_i, \mathcal{A}_i)$ $i = 1, \dots, n$

$M := M_1 \times \dots \times M_n$ with product topology.

Then $\mathcal{A} := \{ (U_1 \times \dots \times U_n, u_1 \times \dots \times u_n : (U_i, u_i) \in \mathcal{A}_i \}$

is C^∞-atlas on M and $pr_i : M \to M_i$ smooth. ✓.

The product M has the following universal property:
If N is a manifold and $f_i : N \to M_i$ $i = 1, .., n$ C^∞-maps,
then $\exists!$ C^∞-map $f : N \to M$ s.t. $pr_i \circ f = f_i$
This characterizes C^∞-structure on M uniquely.

<u>Ex</u> (Disjoint unions) $(M_i, \mathcal{A}_i)$ manifolds $i \in I$ ← countable

$$M := \bigsqcup_{i \in I} M_i := \bigcup_{i \in I} \{(x, i) : x \in M_i\}$$

Equipp it with disjoint union topology and denote by $inj_i : M_i \to M$ $\quad inj_i(x) = (x, i)$.

canonical inclusions.

Then $\mathcal{A} := \bigcup_{i \in I} \mathcal{A}_i$ defines atlas on M, making $inj_i : M_i \to M$ smooth.

Universal property: For any mfd. N and smooth maps $f_i : M_i \to N$ $\exists!$ C^∞-map $f : M \longrightarrow N$ s.t. $f \circ inj_i = f_i$

Property characterizes C^∞-structure on disjoint union.