<u>Recall</u> : $\boxed{\text{Frobenius Thm.}}$

- $E \subseteq TM$ smooth distribution that is involutive.

  $\implies E$ is integrable.

- E involutive distr. $\longleftrightarrow$ $\mathcal{F}_E$ foliation.

Some applications of Thm. 3.38 (Frobenius Thm.) to the study of PDEs :

<u>Ex</u> Consider system of PDEs for a fct. $f : \mathbb{R}^3 \to \mathbb{R}$

(x, y, z) coordinates on $\mathbb{R}^3$

$$(*) \begin{cases} -2z^2 \dfrac{\partial f}{\partial x} + 2x \dfrac{\partial f}{\partial z} = 0 \\ -3z^3 \dfrac{\partial f}{\partial y} + 2y \dfrac{\partial f}{\partial z} = 0 \end{cases}$$

linear System of first order PDEs

Does $(*)$ has any non-constant solutions f ?

$\rightarrow$ see Tutorial.

<u>Ex.</u> Consider system of PDEs for a fct. $f : \mathbb{R}^2 \to \mathbb{R}$

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \alpha(x,y,f(x,y)) \\ \frac{\partial f}{\partial y}(x,y) &= \beta(x,y,f(x,y)) \end{aligned} \qquad (**)$$

α, β smooth fcts defined on an open subset $V \subseteq \mathbb{R}^3$.

<u>Q</u> When does $(**)$ has a solution? Conditions on α, β ?

$\longrightarrow$ see tutorial.

On the opposite ending of integrable distributions (among all distr.) are the so-called bracket-generating distrib.:

<u>Def. 3.41.</u> A smooth distribution $E \subseteq TM$ on a mfd. M is called bracket-generating, if any local frame $\{\xi_1, \dots, \xi_k\}$ of E together with its iterated Lie brackets $[\xi_i, \xi_j]$, $[\xi_i, [\xi_j, \xi_k]]$, ... form a local frame of TM.

<u>Remark</u> If a local frame is bracket generating around some point, then so is another frame around that point.

Ex. Standard contact distribution on $\mathbb{R}^3$

$$E = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \right\rangle$$ smooth rank 2 distribution,

$$\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right] = \frac{\partial}{\partial z}$$ ~~$\notin E$~~ not a section of E.

$T\mathbb{R}^3$ spanned by $\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial z}$ ~~for~~.

Ex. Contact manifolds

M odd dim. mf of dim. $2n+1$. A distribution $E \subseteq TM$ ~~rank~~ is a contact distribution, if E has rank $2n$

s.t. $\mathcal{L}_x : E_x \times E_x \longrightarrow T_xM/E_x \cong \mathbb{R}$ is non-degenerate

$\qquad (\xi, \eta) \longmapsto q_x\left([\tilde{\xi}, \tilde{\eta}](x)\right) \qquad \forall x \in M.$

Levi-bracket

Here, $\tilde{\xi}, \tilde{\eta}$ are extension of ξ, η to local v.f. around x

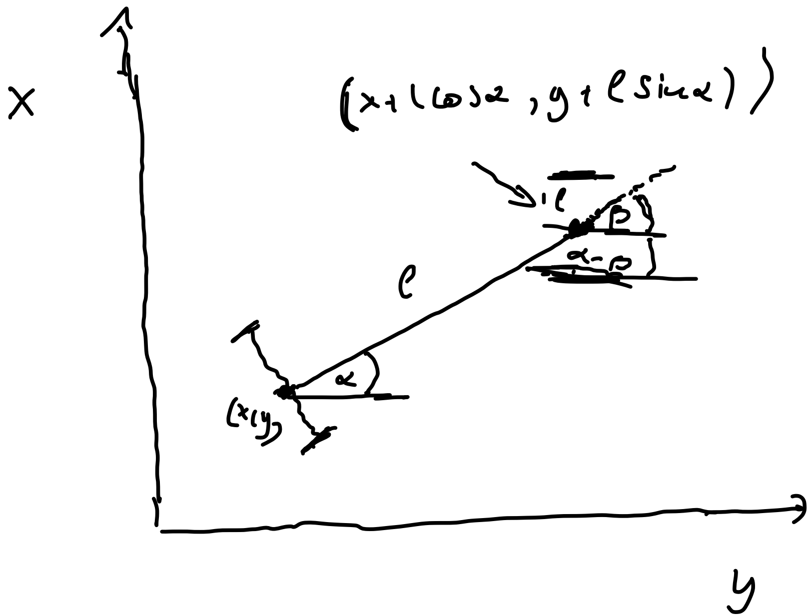
and $q_x : T_xM \longrightarrow T_xM/E_x$ is the standard projection.

Def. A odd dim. mfd. equipped with a contact distribution is called a contact mfd.

$\rightsquigarrow$ contact geometry/topology.

Ex. Driving a car.

Configuration space / phase space of a car: $M = \mathbb{R}^2 \times S^1 \times S^1$
(x, y, α, β)



(x,y) position of midpoint of rear axle

α angle of the chassis to x-axis

β steering angle of the front wheels.

Moving the car traverses a curve $c(t) = (x(t), y(t), \alpha(t), \beta(t))$ in M.

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \text{ is parallel to } \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) + \ell\cos(\alpha(t)) \\ y(t) + \ell\sin(\alpha(t)) \end{pmatrix} \text{ is parallel to } \begin{pmatrix} \cos(\alpha-\beta) \\ \sin(\alpha-\beta) \end{pmatrix}$$

$$\Updownarrow$$

$$x'(t)\sin\alpha(t) - y'(t)\cos\alpha(t) = 0$$

$$\big(x'(t) - \ell\sin(\alpha(t))\,\alpha'(t)\big)\sin(\alpha(t)-\beta(t)) - \big(y'(t) + \ell\cos\alpha(t)\,\alpha'(t)\big)\cos(\alpha(t)-\beta(t)) = 0$$

m) solution

$$\begin{pmatrix} x'(t) \\ y(t) \\ \alpha'(t) \\ \beta'(t) \end{pmatrix} = \lambda(t) \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\frac{\partial}{\partial \beta}} + \mu(t) \begin{pmatrix} l \cos\alpha(t) \cos\beta(t) \\ l \sin\alpha(t) \cos\beta(t) \\ -\sin\beta(t) \\ 0 \end{pmatrix}$$

steer vf : $X := \frac{\partial}{\partial \beta}$

drive vf : $Y := l \cos\beta \cdot \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y} \right) - \sin\beta \frac{\partial}{\partial \alpha}$.

The two 'control' vector fields X and Y span a rank 2 bracket generating distribution on M. $(X, Y, [X,Y], [Y,[X,Y]])$

(TM is spanned by $X, Y, [X,Y], [Y,[X,Y]]$).

Given $E \subseteq TM$ a smooth involutive distribution, we know that through each point $x \in M$ we have an integral submfd, by FB-Thm.

<u>Q</u> What about maximal integral submfd. through a point?

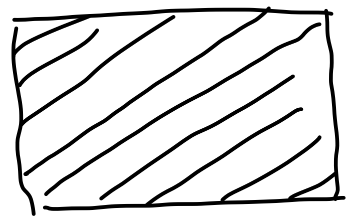
These are in general not submfd. but so called initial submfd.

$$\begin{array}{ccc}
\mathbb{R}^2 & (x,y) & (x,y) \\
 & \downarrow \pi & \downarrow \\
T^2 & & (e^{ix}, e^{iy}) \\
 & (= \mathbb{R}^2/\mathbb{Z}^2) &
\end{array}$$

vector field on $\mathbb{R}^2$,

$$\xi := \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y} \qquad \alpha \in \mathbb{R}$$

Integral curves $\text{const} + t \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$



$\mathbb{R}^2$

ξ is π-related to a v.f. on T^2 $\rightsquigarrow$ integral curves of it are the image of the integral curves of ξ via π.

$$\pi(t(1,\alpha)) = \underline{\underline{(e^{it}, e^{it\alpha})}} \subset T^2$$

If α rational, that's a submfd.

If α is irrational, it's not, because it winds densely around the torus. In appropriate chart around a point $(e^{it}, e^{i\alpha t})$ consists of countably many line segments

<u>Def. 3.4.1</u> M mfd. of dim. n.

1. For a subset $A \subset M$ and $x_0 \in A$ let $C_{x_0}(A) := \{ x \in A : \exists$ a smooth curve $c: [0,1] \to M$ with values in A and $c(0) = x_0$ and $c(1)$ $\}$

(2) $N \subseteq M$ is called an initial submfd. of M of dim. k, if for any $x \in N$ $\exists$ a chart (U, u) for M with $x \in U$ and $u(x) = 0$ and $u(C_x(U \cap N)) = u(U) \cap (\mathbb{R}^k \times \{0\})$.

If $N \subseteq M$ is an initial submfd. Then $\exists!$ C^∞-mfd. structure on N s.t. the inclusion $i : N \hookrightarrow M$ is an injective immersion with the property that for any mfd. P and a map $f : P \to N$ we have f is smooth $\iff$ $i \circ f$ is smooth. $(*)$.

The connected comp. are 2nd countable but uncountably many of them. (so N might be not 2nd countable).

One uses ~~as~~ as an atlas $\mathcal{B} = \{(\underline{C_x(U \cap N)}, u_x)\}_{x \in N}$ for charts as in 2.

- Equipp N with the topology generated by $\underline{C_x(U \cap N)}$ - sets

$\rightsquigarrow$ this topology is in general finer than the subspace topology on N induced from M.

In particular, it is still Hausdorff.

· Transitions maps of Φ are smooth, since restrictions of smooth maps.

· Uniqueness follows from (*) (cf. submf.)

($C_x(U \cap N)$ not open in subspace topolog.; If i is a homeom. onto its image then it is and $C_x(U \cap N) = V \cap N$ for $V \subseteq M$ open and $(V \cap U, \, v|_{V \cap U})$ is a submfd. chart)

Conversely, one may show that the image of an injective immersion $i : N \hookrightarrow M$ with property (*) is an initial submfd.

Coming back to integrable distr. / foliations:

$E \subseteq TM$ ✓integrable with corresp. foliation $\mathcal{F}_E$.

For any $x \in M$ let $\mathcal{F}_x^E := \{ y \in M : \exists\ C^\infty\text{-curve } c : [0,1] \to M$ s.t. $c(0) = x$ and $c(1) = y$ and $c'(t) \in E_{c(t)}\ \forall t \in [0,1] \}$

It is called the <u>leaf through x</u> of $\mathcal{F}_E$.

Note that if a plaque intersects $\mathcal{F}_x^E$ it must be contained in that leaf. Hence, the plaques contained in $\mathcal{F}_x^E$ and restrictions of chart maps can be used to give $\mathcal{F}_x^E$ the structure of a k-dim. mf.

- $i : \mathcal{F}^E_x \hookrightarrow M$ is initial submfd. (Haefl. + AAZ).
- It is an integral submf. ($T_y \mathcal{F}^E_x = E_y \xrightarrow{T_y i} T_y M \quad \forall y$).
- Any connected integral (initial) submf. that intersects $\mathcal{F}^E_x$ is contained in $\mathcal{F}^E_x$ ($\mathcal{F}^E_x$ = " maximal integr. submf. through x ".)

Foliation $\mathcal{F}^E$ devides M into k-dim. initial submfds.

Remark We can equipp M with a differentiable mfd. str. M_E where atlas given by $pr_1 \circ u_\alpha : u_\alpha^{-1}(W_\alpha \times \{a\}) \to W_\alpha \subseteq \mathbb{R}^k$ for $(U_\alpha, u_\alpha) \in \mathcal{F}E$.

Topology on M_E finer than on M and $\mathrm{Id} : M_E \to M$ is injective immersion.

Non holonomic constraints : constraints on position and velocity that can not be integrated to constraints on position only: