Last week:

- $M \subseteq \mathbb{R}^n$ submfd w/ TM tangent space
- $TM \xrightarrow{p} M$ tangent bundle ; sections of $p \rightleftarrows$ vector fields
- local coordinate expressions of vector field.
- $f: M \to N$ a local diffeom. , then we defined $f^*: \mathfrak{X}(N) \to \mathfrak{X}(M)$.

Def. 3.18 $M \subseteq \mathbb{R}^n$ submfd, $\xi \in \mathfrak{X}(M)$, $I \subseteq \mathbb{R}$ interval.

A (smooth) curve $c: I \longrightarrow M$ is called an integral curve of ξ,

if $\quad c'(t) = \xi(c(t))$.

[If $M = U \subseteq \mathbb{R}^n$ open subset : $c'(t) = \xi(c(t))$ system of ordinary differential eq. of first order).

Via Charts Thm. of existence and uniqueness of solutions of a system of first order ODEs implies:

<u>Thm 3.19</u> $M \subseteq \mathbb{R}^n$ submfd, $\xi \in \mathfrak{X}(M)$.

① For any $x \in M$ $\exists!$ maximal integral curve $c_x : I_x \longrightarrow M$, where $I_x \subseteq \mathbb{R}$ interval with $0 \in I_x$ and $c(0) = x$.

② $D(\xi) := \{ (t,x) \in \mathbb{R} \times M : t \in I_x \} \subseteq \mathbb{R} \times M$ is an open subset containing $\{0\} \times M$ and

$$Fl^{\xi} : D(\xi) \longrightarrow M, \quad (t,x) \longmapsto c_x(t)$$

is smooth. The map Fl^{ξ} is called the local flow of ξ.

③ If $y := Fl^\xi(s,x)$ exists, then $Fl^\xi(t+s,x)$ exists $\iff$ $Fl^\xi(t,y)$ exists.

In this case, $Fl^\xi(t+s,x) = Fl^\xi(t, Fl^\xi(s,x))$ $(*)$

In particular, if $D(\xi) = M \times \mathbb{R}$, then $(*)$ says that

$$t \longmapsto Fl^\xi(t, \_) =: Fl^\xi_t$$

$$(\mathbb{R}, +) \longrightarrow (\mathrm{Diff}(M), \circ) .$$

is a group homomorphism.

<u>Notation</u>: $Fl^\xi_t(x) := Fl^\xi(t,x)$.

- Note that (2) and (3) say that for any $x \in M$ $\exists$ an open neighbhd U of x in M and $\varepsilon > 0$ s.t. $Fl^{\xi} : (-\varepsilon, \varepsilon) \times U \to M$ is defined and is a local diffeomorphism for all $|t| < \varepsilon$.

- Note that $Fl_t^{\xi\,*}\xi = \xi$ whenever defined: $T_x Fl_t^{\xi}\, \xi(x) = \xi(Fl_t^{\xi}(x))$.

<u>Def. 3.20</u> $M \subseteq \mathbb{R}^n$ submfd., $\xi \in \mathfrak{X}(M)$.

ξ is called complete, if $D(\xi) = M \times \mathbb{R}$ and hence $Fl^{\xi} : \mathbb{R} \times M \to M$.

Prop. 3.21 $M \subseteq \mathbb{R}^n$ submfd., $\xi \in \mathfrak{X}(M)$.

① Suppose $\exists \varepsilon > 0$ s.t. for any $x \in M$ $\exists$ an open neighbhd. U_x of x in M s.t. the local flow of ξ is defined on $Fl^{\xi} : (-2\varepsilon, 2\varepsilon) \times U_x \to M$.

Then ξ is complete.

② If M is compact, ξ is complete.

Proof

① $$\Psi_t = \left(Fl^{\xi}_{\varepsilon}\right)^{\circ k} \circ Fl^{\xi}_{t-k\varepsilon} = \underbrace{Fl^{\xi}_{\varepsilon} \circ \dots \circ Fl^{\xi}_{\varepsilon}}_{k} \circ Fl^{\xi}_{t-k\varepsilon}$$

k integer part of t/ε; defined $\forall x \in M$ and $t \in \mathbb{R}$.

By ③ of Thm. 3.19, $\varphi_t = Fl_t^\xi$.

② For all $x \in M$ $\exists\, \varepsilon_x > 0$ and a neighbhd. U_x of x in M s.t. $Fl^\xi : (-2\varepsilon_x, 2\varepsilon_x) \times U_x \to M$ is defined.

By compactness of M, there exist $x_1, \dots, x_m \in M$ s.t. $M = U_{x_1} \cup \dots \cup U_{x_m}$. Then $\varepsilon := \min_{i=1,\dots,m} \varepsilon_{x_i}$ satisfies the assumptions of ①.

Ex. $M = \mathbb{R}^2$ $\quad (x,y)$ coordinates on $\mathbb{R}^2$ $\rightsquigarrow \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$$\xi(x,y) = y\frac{\partial}{\partial x} \qquad \eta(x,y) = \frac{x^2}{2}\frac{\partial}{\partial y}$$

$$Fl^{\xi}(t,(x,y)) = (x+ty,\ y) \qquad \left.\frac{d}{dt}\right|_{t=0} (Fl^{\xi}(t,(x,y))) = (y,0)$$

$$= y\frac{\partial}{\partial x}(x,y).$$

$$Fl^{\eta}(t,(x,y)) = \left(x,\ y + t\frac{x^2}{2}\right).$$

ξ and η are complete, but $\xi+\eta$ is not:

recall $\quad (\xi+\eta)(x,y) := y\frac{\partial}{\partial x}(x,y) + \frac{x^2}{2}\frac{\partial}{\partial y}(x,y)$

Integral curve: $c(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad c'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ \frac{x(t)^2}{2} \end{pmatrix}$

$$\Longrightarrow \quad x''(t) = \frac{x(t)^2}{2} \quad \Longrightarrow \quad x'(t)^2 - \frac{x(t)^3}{3} + \text{const.}$$

Solve this for initial value $(y_0^2 - x_0^3)/3 = 0$, $x_0 >$

Integral curve are not defined $\forall t$.

3.4 Tangent vectors as derivations

$M \subseteq \mathbb{R}^n$ submfd.

Def. 3.22 A map $\partial : C^\infty(M,\mathbb{R}) \to \mathbb{R}$ is called a derivation at $x \in M$, if ∂ is $\mathbb{R}$-linear and $\partial(fg) = (\partial f)\, g(x) + f(x)\, \partial g$ $\forall f, g \in C^\infty(M,\mathbb{R})$.

Notation: $Der_x(C^\infty(M,\mathbb{R}), \mathbb{R}) := \{\partial : C^\infty(M,\mathbb{R}) \to \mathbb{R} : \partial \text{ is deriv. at } x\}$

This is a real vector space in the obvious way.

<u>Lemma 3.23</u> $\partial \in \mathrm{Der}_x(C^\infty(M,\mathbb{R}),\mathbb{R})$.

(1) $\partial(1) = 0$ (which implies that $\partial(f) = 0$ for all constant functions f by linearity of ∂).

(2) If $f_1, f_2 \in C^\infty(M,\mathbb{R})$ coincide on an open neighbhd of x in M, then $\partial(f_1) = \partial(f_2)$.

<u>Proof.</u>

(1) $\underline{\partial(1)} = \partial(1\cdot 1) = 1\cdot\partial(1) + \partial(1)\,1 = \underline{2\,\partial(1)}$

$$\Longrightarrow \partial(1) = 0.$$

② Suppose $U \subseteq$ is an open neighborhood of x and $f_1, f_2 \in C^\infty(M,\mathbb{R})$ coincide on U.

Then $f := f_1 - f_2$ vanishes on U. By Cor. 2.32,

$\exists\, g \in C^\infty(M,\mathbb{R})$ s.t. $\operatorname{supp}(g) \subseteq U$ and $g(x) = 1$.

$$0 = \partial(0) \underset{\substack{\uparrow \\ \text{since} \\ \operatorname{supp}(g) \subseteq U \\ \text{and } f|_U = 0}}{=} \partial(f \cdot g) = \partial f \underbrace{g(x)}_{=1} + \underbrace{f(x)}_{=0} \partial g = \partial f = \partial(f_1) - \partial(f_2).$$

Any tangent vector $\xi \in T_x M$ induces a derivation at x:

$$\partial_\xi : \underset{\underset{C^\infty(M,\mathbb{R})}{\cap}}{f} \longmapsto T_x f \xi \in T_{f(x)}\mathbb{R} \simeq \mathbb{R}$$

We also write $\partial_\xi(f) := \xi \cdot f$

Indeed, let $c : I \to M$ C^∞- curve with $c(0) = x$ and $c'(0) = \xi$ and $f, g \in C^\infty(M, \mathbb{R})$, $\lambda \in \mathbb{R}$. Then:

$$(f + \lambda g) \circ c = f \circ c + \lambda (g \circ c) \quad (*)$$

$$\Rightarrow \partial_\xi (f + \lambda g) = ((f + \lambda g) \circ c)'(0) = (f \circ c)'(0) + \lambda (g \circ c)'(0)$$

$$= \partial_\xi (f) + \lambda \partial_\xi (g)$$

$\Rightarrow \partial_\xi : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ is $\mathbb{R}$-linear.

Also $fg \circ c = (f \circ c)(g \circ c)$ hence

$$((fg)\circ c)'(0) = (f\circ c)'(0)\, g(x) + f(x)\,(g\circ c)'(0) =$$
$$= \partial_\xi(f)\, g(x) + f(x)\, \partial_\xi(g).$$

$\Longrightarrow$ ∂_ξ is derivation at x for any $\xi \in T_x M$.

Let (U, u) be a chart for M with $x \in U$.

$\leadsto$ $\frac{\partial}{\partial u^1}(x), \ldots, \frac{\partial}{\partial u^k}(x) \in T_x M$ form basis of $T_x M$.

$$\frac{\partial}{\partial u^i}(x) \cdot f = T_x f\, T_{u(x)} u^{-1}(u(x), e_i) = T_{u(x)}(f \circ u^{-1})(u(x), e_i)$$

$= \left(f(x), D_{u(x)} (f \circ u^{-1}) e_i \right)$. ← i-th partial derivative at $u(x)$ of the local coordinate expression ~~of~~ $f \circ u^{-1}$: $u(U) \to \mathbb{R}$ of f.

We write $\frac{\partial}{\partial u^i}(x) \cdot f =: \underline{\frac{\partial f}{\partial u^i}(x)}$

Since any $\xi_x \in T_x M$ can be written

as $\xi = \sum_{i=1}^{k} \xi^i \frac{\partial}{\partial u^i}(x)$ $\quad \xi^i \in \mathbb{R}$.

we have

$$\partial_\xi(f) = T_x f \xi = \sum_{i=1}^{k} \xi^i \, T_x f \frac{\partial}{\partial u^i}(x) = \sum_{i=1}^{k} \xi^i \frac{\partial f}{\partial u^i}(x).$$

Thm 3.24 $M \subseteq \mathbb{R}^n$ submfd, $x \in M$. The map

$$\Psi_x : T_x M \longrightarrow \operatorname{Der}_x(C^\infty(M,\mathbb{R}),\mathbb{R})$$
$$\xi \longmapsto \partial_\xi$$

is a linear isomorphism. Moreover, the following diagram commutes

$$(*) \quad \begin{tikzcd} T_x M \arrow[r, "\Psi_x", "\sim"'] \arrow[d, "T_x F"'] & \operatorname{Der}_x(C^\infty(M,\mathbb{R}),\mathbb{R}) \arrow[d, "F_*"] \\ T_{F(x)} N \arrow[r, "\Psi_{F(x)}"] & \operatorname{Der}_{F(x)}(C^\infty(N,\mathbb{R}),\mathbb{R}) \end{tikzcd}$$

$F_*(\partial)(g) := \partial(g \circ F)$ $\forall g \in C^\infty(N,\mathbb{R})$

for any C^∞-map $F : M \to N$ and $N \subseteq \mathbb{R}^m$ submfd.

Proof

- Linearity of φ_x: ✓, since $T_x f$ is linear for any $f \in C^\infty(M,\mathbb{R})$.
- Commutativity of $(*)$: $F_*(\varphi_x(\xi))(g) =$

$$= F_*(\partial_\xi)(g) = \partial_\xi(g \circ F) =$$

$$= T_x(g \circ F)\xi = T_{F(x)} g \circ T_x F \xi =$$

$$= \partial_{T_x F \xi}(g)$$

- Injectivity of φ_x: If $\underset{\neq 0}{\xi} \in T_x M$, then

$\partial_\xi \neq 0$ (i.e. $\exists f \in C^\infty(M,\mathbb{R})$ s.t. $\partial_\xi(f) \neq 0$).

Let V be an open neighbhd. of x s.t. $\overline{V} \subset U$ and (U,u) a chart. By Cor. 2.32, $\exists$ C^∞-fct.

$g \in C^\infty(M,\mathbb{R})$ with $\operatorname{supp}(g) \subseteq U$ and $g|_{\bar{V}} \equiv 1$.

Then $g \cdot u^i$ can be extended by zero to C^∞-fct. $\tilde{u}^i : M \to \mathbb{R}$.

By construction, $\tilde{u}^i \circ u^{-1}$ locally around $u(x)$ equals the i-th projection $u(U) \subseteq \mathbb{R}^k \to \mathbb{R}$

If $\xi = \sum_{i=1}^{k} \xi^i \frac{\partial}{\partial u^i}(x)$, then $\partial_\xi(\tilde{u}^i) = \sum_{j=1}^{k} \xi^j \underbrace{\frac{\partial \tilde{u}^i}{\partial u^j}(x)}_{= 0 \text{ unless } i=j} = \xi^i$

<u>Surjectivity of φ_x:</u> (U,u) chart, $x \in U$. We may assume $u(x) = 0$ and $u(U) \supset B_1(0) := \{z \in \mathbb{R}^k : \|z\| < 1\}$.

If $y \in U$ s.t. $u(y) \in B_1(0)$, then for $f \in C^\infty(M, \mathbb{R})$ we have

$$f(y) = f(x) + \int_0^1 \frac{d}{dt}\Big|_{t=0} (f \circ u^{-1})(t u(y))\, dt.$$

$$= f(x) + \int_0^1 \sum_i \frac{\partial (f \circ u^{-1})}{\partial x^i}(t u(y))\, \underline{u^i(y)}\, dt$$

$$= f(x) + \sum_i u^i(y) \underbrace{\int_0^1 \frac{\partial (f \circ u^{-1})}{\partial x^i}(t u(y))\, dt}_{=: h_i(y)}.$$

$$h_i : u^{-1}(B_1(0)) \longrightarrow \mathbb{R}.$$