$E \subseteq TM$ integrable distribution.

$\rightsquigarrow$ $\mathcal{F}^E$ foliation

For any $x \in M$, $\mathcal{F}^E_x := \{ y \in M : \exists\ C^\infty\text{-curve } c : [0,1] \to M$ s.t. $c(0) = x$ and $c(1) = y$ and $c'(t) \in E_{c(t)}, \forall t \}$

leaf through x of $\mathcal{F}^E$.

- $i : \mathcal{F}^E_x \hookrightarrow M$ initial submf., integral (initial) submf. of E.

- Any connected integral (initial) submfd. that intersects $\mathcal{F}^E_x$ is contained in $\mathcal{F}^E_x$ (maximality).

Foliation $\mathcal{F}^E$ devides M into k-dim. initial submfds.

<u>Remark 3.43</u> M set equipped with C^∞-atlas $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$

- $M = \bigcup_{\alpha \in I} U_\alpha$, $u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subseteq \mathbb{R}^n$, trans. maps smooth.

subset of M — open subset of $\mathbb{R}^n$

$\exists!$ topology on M for which $U_\alpha \subseteq M$ are open and u_α are homeomorphism. It's the coarsest topology making the charts homeomorphism. (i.e. $V \subseteq M$ open $\Leftrightarrow$ $u_\alpha(U_\alpha \cap V)$ is open in $u_\alpha(U_\alpha)$ $\forall$ charts (U_α, u_α)).

Remark 3.44

Assume M is a mfd. with an integrable distr. E (or equiv. foliation $\mathcal{F}^E$).

We may equipp M with different mfd. structure M_E where the atlas $\mathcal{A}_E$ is given by $pr_1 \circ u_\alpha : u_\alpha^{-1}(W_\alpha \times \{a\}) \to W_\alpha \subseteq \mathbb{R}^k$

By Remark 3.42, $\exists$ topology on M_E making $(M_E, \mathcal{A}_E)$ a smooth k-dim. mfd. (topology on M_E is finer than the one on M).

Connected components of M_E are the leaves of $\mathcal{F}^E$.

M_E is Hausdorff but has uncountably many connected components. (

Global Frobenius Thm :

Thm. 3.45 M mfd. , $E \subseteq TM$ smooth involutive distribution of rank k and $\mathcal{F}^E$ the corresp. foliation.

- If $E \neq TM$, the topology on M_E is finer as that of M.
- M_E has uncountably many connected components given by the leaves of $\mathcal{F}^E$.
- $\mathrm{Id} : M_E \to M$ bijective smooth immersion and each leaf of $\mathcal{F}$ is an initial submf. of M.

4. The Cotangent Bundle

Constructions / operations in the category of vector spaces can be generalized to the category of vector bundles. In particular, for a vector bundle we can form its dual and take wedge products of it.

4.1 1-Forms

- M mfd. of dim. n.
- $E \xrightarrow{p} M$ vector bundle of rank k.

Given two trivializations of E, $\phi_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ and $\phi_\beta : p^{-1}(U_\beta) \longrightarrow U_\beta \times \mathbb{R}^k$, the transition map is of the form,

$$\phi_\beta \circ \phi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k$$
$$(y, v) \longmapsto (y \ \phi_{\beta\alpha}(y) v)$$

for a unique smooth $\phi_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R})$.

<u>Remark</u> 1 Trivializations of E form a so-called vector bundle atlas of E.

2 A vector bundle may be also defined as a mf E together with

a smooth map $p: E \to M$ and a maximal vector bundle atlas.

, The family of maps $(\phi_{\beta\alpha})_{\alpha,\beta \in I}$ satisfy

$$\phi_{\alpha\alpha}(x) = id$$

$$\phi_{\alpha\beta}(x) \circ \phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x)$$

cocycle condition.

Cohomology class of cocycle of trans. fcts. determines vector bundle up to isomorphism.

For any $x \in M$ consider $E_x^* = \{ \lambda : E_x \to \mathbb{R} \ : \ \lambda \text{ linear} \}$, the dual vectorspace of E_x.

$$\begin{array}{c} E^* := \bigsqcup_{x \in M} E_x^* \\ q \big\downarrow \\ M \end{array} \qquad \text{natural projection.}$$

Claim / Def. 4.1 $E^* \xrightarrow{q} M$ is again a vector bundle of rank k over M. It is called the dual vector bundle of E.

<u>Proof</u> $E^* \xrightarrow{q} M$ surjection, $q^{-1}(x) = E^*_x$ is a vector space for $x \in M$.

Fix $x \in M$ and let (U_α, u_α) be a chart for M with $x \in U_\alpha$.

By possibly shrinking U_α, we may assume E trivializes over U_α:

$\exists$ a diffeom. s.t.

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\Phi_\alpha", "\sim"'] \arrow[dr, "p"'] & & U_\alpha \times \mathbb{R}^k \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

and $\Phi_\alpha\big|_{E_y} : E_y \to \{y\} \times \mathbb{R}^k$ lin. isomorphism $\forall y \in U_\alpha$.

Define bijections
$$\begin{tikzcd} q^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha^*"] \arrow[dr, "q"'] & & U_\alpha \times (\mathbb{R}^k)^* \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

by $\phi_\alpha^*\big|_{E_y^*} := \left(\phi_\alpha\big|_{E_y}^{-1}\right)^* : E_y^* \to \{y\} \times (\mathbb{R}^k)^*$.

Then $(u_\alpha \times \mu) \circ \phi_\alpha^* : q^{-1}(U_\alpha) \to \underbrace{u_\alpha(U_\alpha)}_{\subseteq \mathbb{R}^n} \times \mathbb{R}^k \subseteq \mathbb{R}^{n+k}$

is a bijection for any choice of isomorphism, $\mu : \mathbb{R}^{k*} \xrightarrow{\sim} \mathbb{R}^k$.

For another such bijection $(u_\beta \times \mu) \circ \phi_\beta^*$ we have

$$\left((u_\beta \times \mu) \circ \phi_\beta^*\right) \circ \left(u_\alpha \times \mu \circ \phi_\alpha^*\right)^{-1} : u_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

$$\longrightarrow u_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

$$(y, v) \mapsto \left(u_\beta \circ u_\alpha^{-1}(y), \mu \circ \underline{\left(\left(\phi_{\beta\alpha}(u_\alpha^{-1}(y))\right)^*\right)^{-1}} \mu^{-1}(v)\right).$$

which is smooth, since $u_\beta \circ u_\alpha^{-1}$, $\phi_{\beta\alpha}$ and inverse in $GL(k, \mathbb{R})$ are smooth.

By Remark 3.43, we hence can use these bijections to equip E^* with the structure of a C^∞-mfd. of dim $n+k$.

By construction, we also have that $E^* \xrightarrow{q} M$ is a vector bundle over M. (ϕ_α^* are trivializations.).

Def. 4.2

① For any mfld. M the dual vector bundle $q: T^*M \to M$ of $p: TM \to M$ is called cotangent bundle of M. We write $q^{-1}(x) := T_x^*M$.

② A (smooth) local section of q is called a (smooth) local 1-form..

Notation: We write $\Omega^1(M)$ or $\Gamma(T^*M)$ for the the set of 1-forms. As for any vector bundle, $\Gamma(T^*M)$

is a real vector space and a module over the ring $C^\infty(M,\mathbb{R})$.

Suppose (U,u) is a chart for M. Then we have

$$
\begin{tikzcd}
T^*U \arrow[r, "T^*u"] \arrow[dr, "q"'] \arrow[rr, bend left, "\Phi^*"] & u(U) \times \mathbb{R}^{n*} \arrow[r, "u^{-1} \times \mathrm{id}"] & U \times \mathbb{R}^{n*} \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}
$$

$$T^*_y u := T^*u\big|_{T^*_y U}$$

$$T^*_y u := \left((T_y u)^{-1} \right)^*$$

$$T_y u : T_y U \to u(U) \times \mathbb{R}^n$$

Denote by $\{\lambda_1, \dots, \lambda_n\}$ the basis of $\mathbb{R}^{n*}$ dual to the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ $\quad (\lambda_i(e_j) = \delta_{ij})$.

$\exists$ local sections du^i of $T^*U \to U$ defined by

$$\phi^*(du^i(y)) = (y, \lambda_i) \quad \forall y \in U.$$

$$du^i(y) = (\phi^*)^{-1}(y, \lambda_i) = (T^*u)^{-1}(u(y), \lambda_i).$$

For any $y \in U$, $du^1(y), \dots, du^n(y)$ is a basis of T_y^*U.

For any smooth fcts $\omega_i : U \to \mathbb{R}$ $\quad i = 1, \dots, n$, $\quad \sum_{i=1}^n \omega_i du^i$

is again local 1-form defined on U. Hence, locally $\exists$ many 1-forms and via partitions of unity that holds also globally.

Conversely, any $\omega \in \Omega^1(M)$ may be restricted to U and can be uniquely written as

$$\omega\Big|_U = \sum_{i=1}^{n} \omega_i \, du^i \quad \text{for } \omega^i \in C^\infty(U, \mathbb{R}), \; i = 1, \dots, n.$$

$$\left(T^*u \circ \omega \circ u^{-1} : u(y) \mapsto \left(u(y), \sum_{i=1}^{n} \omega_i(y) \lambda_i \right) \right)$$

Def. 4.3 $\omega \in \Omega^1(M) = \Gamma(T^*M)$ and (U,u) a chart for M.

Then $$\omega\big|_U = \sum_{i=1}^n \omega_i \, du^i \quad ; \quad \omega_i \in C^\infty(U,\mathbb{R}).$$

and $(\omega_1, \dots, \omega_n)$ is called the local coordinate expression of ω w.r.t. (U,u).

$V \qquad V^* \qquad V \xrightarrow{\simeq} V^{**}$

Note that we have a bilinear map:

$$V \times V^* \longrightarrow \mathbb{R}$$
$$(v,\lambda) \longmapsto \lambda(v) \in \mathbb{R}$$

$$\Gamma(T^*M) \times \Gamma(TM) \longrightarrow C^\infty(M,\mathbb{R})$$
$$(\omega, \xi) \longmapsto \big(\omega(\xi) : x \longmapsto \underline{\omega_x(\xi_x)}\big)$$

By construction, $du^i\left(\frac{\partial}{\partial u^j}\right)(y) = \delta_{ij} \quad \forall y \in U$

and hence $\omega|_U\left(\frac{\partial}{\partial u^i}\right) = \omega_i$ and $du^i\left(\xi|_U\right) = \xi^i$

Remark. For ω not necessarily smooth section of T^*M, then the following are equiv.:

1. ω is smooth.
2. ω has smooth local coordinate express. for any chart (U, u).
3. $\omega(\xi) \in$ is smooth fct. for any smooth vector field ξ.

Coordinate change: (U_α, u_α) and (U_β, u_β) two charts.

and $\omega \in \Omega^1(M)$. Recall $\displaystyle \frac{\partial}{\partial u_\alpha^i} = \sum_{j=1}^{n} A_i^j \frac{\partial}{\partial u_\beta^j}$

where $\displaystyle A_i^j = \frac{\partial u^j_{\beta\alpha}}{\partial y^i}$. $\Longrightarrow$ $\displaystyle \omega\Big|_{U_\alpha \cap U_\beta} = \sum \omega_i^\alpha \, du_\alpha^i = \sum \omega_i^\beta \, du_\beta^i$

where $\displaystyle \omega_i^\alpha = \omega\left(\frac{\partial}{\partial u_\alpha^i}\right) = \sum_{j=1}^{n} A_i^j \, \omega\left(\frac{\partial}{\partial u_\beta^j}\right) = \sum_{j=1}^{n} A_i^j \omega_j^\beta$.

resp. $\displaystyle \omega_i^\beta = \sum_j B_i^j \omega_j^\alpha$ where B_i^j is the inverse of A_j^i.

If $f \in C^\infty(M,\mathbb{R})$, then we define 1-form $df \in \Omega^1(M)$ by

$$df(x)(\xi_x) := T_x f\, \xi_x \quad . \quad \forall x \in M,\ \xi_x \in T_x M .$$

$$(T_x f : T_x M \to \mathbb{R}).$$

[$df : M \to T^*M$ is smooth, since $df(\xi) = \xi \cdot f \in C^\infty(M,\mathbb{R})$. $\forall \xi \in \Gamma(TM)$ and $df(x) \in T^*_x M \quad \forall x \in M$].

The operator $d : C^\infty(M,\mathbb{R}) \longrightarrow \Omega^1(M)$ is the easiest special case of the so-called exterior derivative on diff. forms.

In local coordinates (U, u) we have:

$$df\Big|_U = \sum_{i=1}^{n} df\left(\frac{\partial}{\partial u^i}\right) du^i = \sum_i \frac{\partial f}{\partial u^i} du^i .$$

i-th par. deriv. of $f \circ u^{-1}$

Note that for $f = u^i$ one of the coord. fct.

we $du^i = \sum du^i\left(\frac{\partial}{\partial u^j}\right) du^j = du^i$ ✓

which justifies our notation for the du^i's.

$V \qquad V^*$

$\boxed{V \times V^* \to \mathbb{R}}$ $\quad V \simeq V^{**}$.

$(v, \lambda) \mapsto \lambda(v)$

$\langle\ ,\ \rangle$ fix ... ... on V

$\boxed{V \times V \to \mathbb{R}} \rightsquigarrow \boxed{V \simeq V^*}$

M

$g_x : T_xM \times T_xM \to \mathbb{R}$ $\qquad df \longleftrightarrow$

$\underline{g} : \underline{TM} \simeq \underline{T^*M}$ $\quad \# \quad \xi \longmapsto \underline{g}(\xi, -)$ $\in \Omega^1(M)$.