Yesterday: Introduced submfs in $\mathbb{R}^n$

Remark: As a subset of $\mathbb{R}^n$ a submf. $M \subseteq \mathbb{R}^n$ inherits a topology from $\mathbb{R}^n$ (subspace topology):

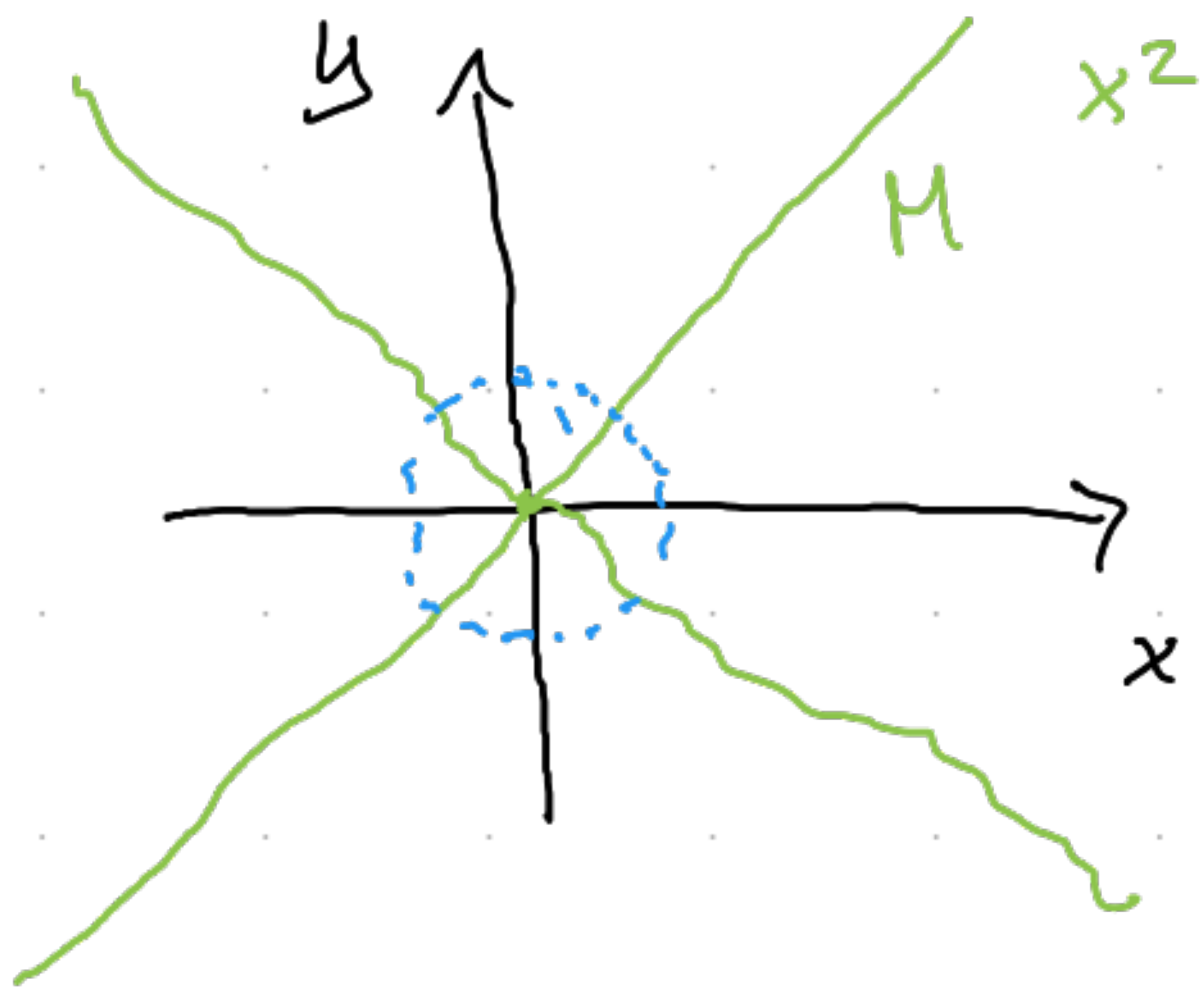
$$U \subseteq M \text{ is open} \iff U = M \cap \tilde{U}, \quad \tilde{U} \subseteq \mathbb{R}^n$$

Remark:

- If one replaces „smooth"/C^∞ by C^r for $1 \leq r \leq \infty$ or C^ω in Def. 2.1 – 2.4 one obtains C^r-submfds and real analytic submfds of $\mathbb{R}^n$.
- If one replaces $\mathbb{R}$ by $\mathbb{C}$ and C^∞ by holomorphic, then obtains complex submfds in $\mathbb{C}^n$.
- Replacing in Def. 2.1 C^∞ by C^0 one obtains topological submfds of $\mathbb{R}^n$. Also Def. 2.2 makes sense for C^0

but it is not equiv. to Def. 1.1.

Examples that are not smooth submfds of $\mathbb{R}^2$:



$x^2 - y^2 = 0$

Problems at $(0,0)$:

$f(x,y) = x^2 - y^2$

has vanishing diff. at $(0,0)$.

$t \mapsto \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ (

Problem at $(0,0)$

Not smooth submf but topolog. one.

Examples of smooth submfds of $\mathbb{R}^n$:

- Any open subset $U \subseteq \mathbb{R}^n$ is an n-dim. submfd of $\mathbb{R}^n$ (take $\mathrm{id}: U \to U$ as param. / trivializ.) and any n-dim. submfd is an open subset.
- Any open subset $U \subseteq \mathbb{R}^n$ can be viewed as an n-dim. submfd of $\mathbb{R}^N$, $N \geq n$ via $\mathbb{R}^n \hookrightarrow \mathbb{R}^N$.
- Any open subset of a submfd of $\mathbb{R}^n$ is again a submfd.
- If $M \subseteq \mathbb{R}^n$ and $N \subseteq \mathbb{R}^m$ submfds of dim. k resp. ℓ, then $M \times N \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is a submfd of dim. $k+\ell$

Now some less trivial examples:

Ex (Spheres)

$\mathbb{R}^{n+1}$ equipped with standard inner product $\langle\, , \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$

$$M := S^n := \{ x \in \mathbb{R}^{n+1} : \|x\| = \sqrt{\langle x,x\rangle} = 1 \} \subseteq \mathbb{R}^{n+1}$$

$n=1$ $\quad S^1 \subset \mathbb{R}^2$ unit circle

It is an n-dim. submfd of $\mathbb{R}^{n+1}$:

• $\boxed{\text{Description by smooth reg. equations:}}$

$$f : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R} \qquad f(x) = \langle x,x\rangle - 1$$

f

$$\rightsquigarrow \quad D_x f(y) = \frac{d}{dt}\Big|_{t=0} f(x+ty) = \frac{d}{dt}\Big|_{t=0} \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle = 2\langle x,y\rangle$$

Graph: $x \in M = S^n$

$$\mathbb{R}^{n+1} = W \oplus W^\perp \qquad W = x^\perp \quad W^\perp = \mathbb{R}x$$

$U := \{ y \in \mathbb{R}^{n+1} : \langle x, y \rangle > 0 \} \subseteq \mathbb{R}^{n+1}$ open neighbhd. of x.

$V := \{ z \in W : \|z\| < 1 \}$

$$g : V \to W^\perp \qquad g(z) := \sqrt{1 - \|z\|^2}\, x$$

$$\mathrm{gr}(g) = \{ (z, g(z)) : z \in V \} = M \cap U$$

$$\|g(z) + z\| = \sqrt{\|z\|^2 + (1 - \|z\|^2) \cdot \underbrace{\|x\|^2}_{=1}} = 1$$

Parametrization: U, V, W, g as above

$$\varphi : V \to W \oplus W^\perp = \mathbb{R}^{n+1}$$

$$\varphi(z) = (z, g(z)).$$

<u>Trivialization:</u> U, V as above.

$$y \in U \qquad \tilde{\phi}(y) = \frac{y}{\|y\|} - \frac{\langle y, x\rangle}{\|y\|} x$$

$$\tilde{\phi} : U \to V$$

$$\tilde{\phi}(y) = \tilde{\phi}(y') \Leftrightarrow y = a y'$$

$$\phi : U \longrightarrow V \times \{\lambda x : \lambda > -1\} \subseteq \mathbb{R}^{n+1} = W \oplus W^{\perp}$$

$$\phi(y) = \left(\tilde{\phi}(y), (\langle y, y\rangle - 1) x\right) \quad \text{smooth}$$

It is diffeom. with inverse

$$\phi^{-1} : (z, \lambda x) \longmapsto \sqrt{\lambda + 1}\left(z + (1 - \langle z, z\rangle) x\right)$$

$$\phi(U \cap M) = V \times \{0\} = V \times \{\lambda x : \lambda > -1\} \cap W.$$

Ex. (Hyperboloids, Ellipsoids)

Fix $a_1, \dots, a_{n+1} \in \mathbb{R}_{>0}$

$$f : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}$$

$$f(x_1, \dots, x_{n+1}) = \sum_{i=1}^{\ell} \frac{x_i^2}{a_i^2} - \sum_{i=\ell+1}^{n+1} \frac{x_i^2}{a_i^2} - 1$$

f is reg. smooth fct. and so $M := f^{-1}(0)$ is an n-dim. submf. of $\mathbb{R}^{n+1}$.

Ex. (Torus)

$\mathbb{C}^n \cong \mathbb{R}^{2n}$ as real vector space

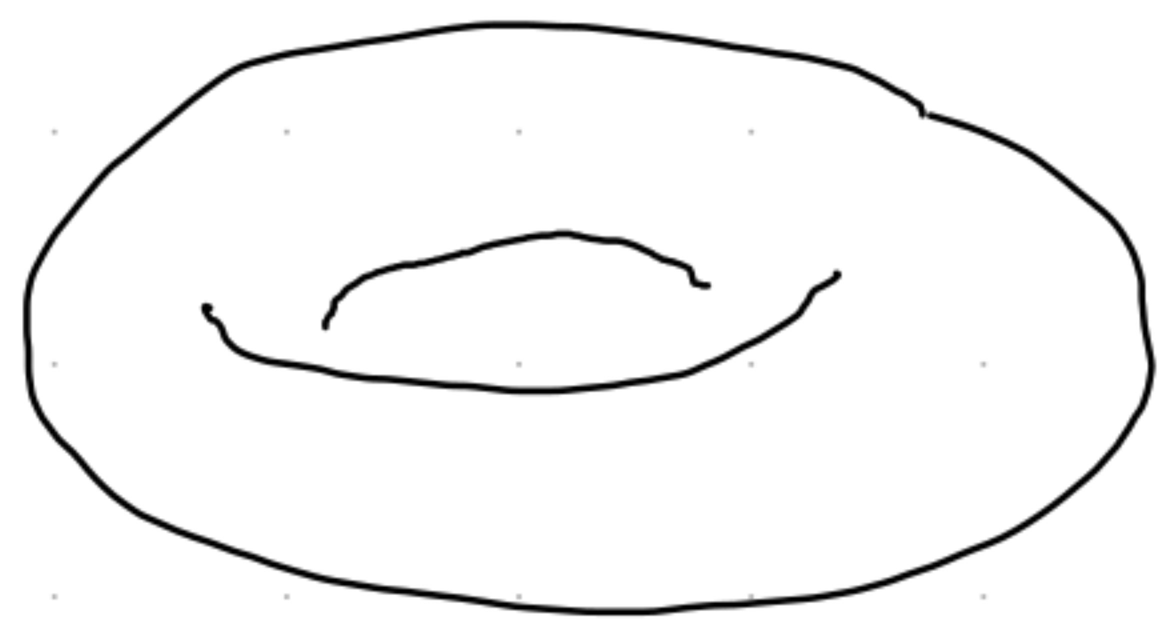
$$T^n := \{ z \in \mathbb{C}^n : |z_1| = \dots = |z_n| = 1 \} \subseteq \mathbb{R}^{2n}$$

$f : \underbrace{\mathbb{R}^{2n} \setminus \{0\}}_{\cong \mathbb{C}^n \setminus \{0\}} \longrightarrow \mathbb{R}^n \quad f(z_1, \dots, z_n) = (|z_1|^2 - 1, \dots, |z_n|^2 - 1)$

$\Rightarrow T^n = f^{-1}(0) \subseteq \mathbb{R}^{2n}$ n-dim. Subf. in $\mathbb{R}^{2n}$

$$T^n = \underbrace{S^1 \times \ldots \times S^1}_{n} \subseteq \mathbb{R}^2 \times \ldots \times \mathbb{R}^2 = \mathbb{R}^{2n}$$

$n = 2$



<u>Ex</u> (Lie groups)

$\mathrm{End}(\mathbb{R}^n)$ = vector space of linear maps $\mathbb{R}^n \to \mathbb{R}^n$

Via a choice of basis of $\mathbb{R}^n$: $\mathrm{End}(\mathbb{R}^n) \cong M_n(\mathbb{R})$
$= \{n \times n$ real matrices$\}$
$\cong \mathbb{R}^{n^2}$

$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous

$\Longrightarrow$ $GL(n,\mathbb{R}) := \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}$

$\subseteq M_n(\mathbb{R})$ is open subset

$\rightarrow$ with respect to matrix multipl. it is also a group, called the general linear group.

In fact, $\det : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}$ smooth and regular and so is also $f := \det - 1$. $f(A) = \det(A) - 1$

$$D_A f(A) = \frac{d}{dt}\Big|_{t=0} \underbrace{\det(A + tA)}_{\det((1+t)A)} = \frac{d}{dt}\Big|_{t=0} (1+t)^n \det(A)$$

$$= n \det(A) \neq 0$$

$\Rightarrow$ $SL(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) : \det(A) = 1\} = f^{-1}(0)$
$\subseteq GL(n,\mathbb{R}) \subseteq M_n(\mathbb{R})$ is an (n^2-1)-dim. submfd. and a subgroup of $GL(n,\mathbb{R})$
$(\det(AB) = \det(A)\det(B))$.

called the special liner group.

Now consider $f: GL(n,\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$$f(A) = A \cdot A^t - Id_n$$

Symmetric $n \times n$ matrices.

$f(A)^t = f(A) \Rightarrow f: GL(n,\mathbb{R}) \longrightarrow M_n^{sym}(\mathbb{R}) \simeq \mathbb{R}^{\frac{n(n+1)}{2}}$

$$O(n) := O(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) : A^{-1} = A^t\}$$
$$= f^{-1}(0).$$

f is smooth and also regular:

$F: (A,B) \longmapsto A \cdot B^t$ bilinear

$$D_A f(B) = \frac{d}{dt}\Big|_{t=0} F(A+tB, A+tB) = \frac{d}{dt}\Big|_{t=0} \big(F(A,A) + tF(A,B) + tF(B,A) + t^2 F(B,B)\big)$$

$$= A \cdot B^t + B \cdot A^t$$

If $S \in M_n^{sym}(\mathbb{R})$, $A \in O(n)$, then set

$$B := \frac{1}{2} SA.$$

$$\Rightarrow D_A f\, B = \frac{1}{2}\Big(A \underbrace{(SA)^t}_{\substack{A^t . S^t \\ = A^t . S}} + S \underbrace{A A^t}_{Id} \Big) = S$$

$$\underbrace{A^t . S^t = A^t . S}_{S} \qquad \underbrace{Id}_{= S}$$

$\Rightarrow O(n)$ is a submfd. ~~of $\mathbb{R}^{n^2}$~~ of dim. $\frac{n(n-1)}{2}$.

It is also a subgroup of $GL(n, \mathbb{R})$,

called the orthogonal group.