A few comments:

(1) Corrections:

- L_X, R_X and their flows

$$Fl_t^{L_X}(B) = \underline{B e^{tX}} \qquad \lambda_B^* L_X = L_X \quad \forall B \in GL(n,\mathbb{R})$$

$$Fl_t^{R_X}(B) = e^{tX} B \qquad \Longrightarrow Fl_t^{L_X} \circ \lambda_B = \lambda_B \circ Fl_t^{L_X}$$

$$Fl_t^{L_X}(B) = \lambda_B(e^{tX}) = B e^{tX}$$

- Ex:

$$-2z^2 \frac{\partial f}{\partial x} + 2x \frac{\partial f}{\partial z} = 0$$

$$-3z^2 \frac{\partial f}{\partial y} + 2y \frac{\partial f}{\partial z} = 0$$

$\rightsquigarrow$ X, Y $\quad E = \langle X, Y \rangle$ involutive

$\Longrightarrow$ E is spanned by $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$ for the appropriate chart (U, u).

② Clarification :

In ②, $\alpha(x,y,z) = z\cos y$

$$\frac{\partial f}{\partial x}(x,y) = \alpha(x,y,f(x,y))$$

$$\frac{\partial f}{\partial y}(x,y) = \beta(x,y,f(x,y))$$

$\exists$ solution $\iff \frac{\partial}{\partial y}\left(\alpha(x,y,f(x,y))\right) = \frac{\partial}{\partial x}\left(\beta(x,y,f(x,y))\right)$

since $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial y}$.

$(x,y) \xrightarrow{\varphi} (x,y,f(x,y)) \xrightarrow{\alpha} \mathbb{R}$

$$\Longrightarrow \boxed{\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z}}$$

$\alpha : \mathbb{R}^3 \to \mathbb{R}$

$\alpha(x,y,z)$

<u>Def. 4.21</u> A graded derivation of degree r of $(\Omega(M), \wedge)$ is a linear map $D: \Omega(M) \to \Omega(M)$ s.t.

$$\Omega(M) = \bigoplus_{i \geq 0} \Omega^i(M)$$

$D: \Omega^k(M) \to \Omega^{k+r}(M)$ and $D(\omega \wedge \eta) = D(\omega) \wedge \eta + (-1)^{rk} \omega \wedge D\eta$ for $\omega \in \Omega^k(M)$.

<u>Last week:</u> M mfd. of dim. n.

$\omega \in \Omega^n(M)$

$$\omega|_U = \omega^U_{1\dots n} du^1 \wedge \dots \wedge du^n \quad \text{for a chart } (U, u).$$

$$\omega^U_{1\dots n} : U \to \mathbb{R}$$

$$\omega^n\left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}\right).$$

If (V, v) is another chart with $V = U$. Then:

$$\underbrace{\omega^U_{1\dots n} \circ u^{-1}}_{\overset{\parallel\;(*)}{\det(D\phi)\, \underline{\omega^V_{1\dots n} \circ v^{-1} \circ \phi}}} : u(U) \to \mathbb{R}$$

$$\phi : u(U) \to v(U)$$
$$\phi = v \circ u^{-1}$$

$$\Longrightarrow \int_{\underline{v(U)} \subseteq \mathbb{R}^n} \omega^V_{1\dots n} \circ v^{-1} = \int_{u(U)} \omega^V_{1\dots n} \circ v^{-1} \circ \phi \; |\mathrm{Det}\, D\phi|$$

equals up to a sign $\int_{u(U)} \omega^U_{1\dots n} \circ u^{-1}$

Hence, integral of local coordinate expression of ω is up to a sign independent of chosen chart.

5.1. Orientation

Suppose V is an n-dim. vector space.

If $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are ordered basis of V, then $\exists!$ linear map $A: V \to V$ s.t. $A(a_i) = b_i \; \forall i$.

The two basis $\{a_i\}$ and $\{b_i\}$ have the same orientation, if $\det(A) > 0$, if $\det(A) < 0$, they have opposite orientation.

- „Having the same orientation" defines an equivalence relation on the set of ordered basis of V and $\exists$ exactly two equivalence classes.

- An orientation on V is the choice of one of these two classes and a vector space with chosen orientation is called an oriented vector space. Having chosen an orientation on V, we call the ordered basis in this chosen equiv. class ~~is~~ positively oriented and the ones in the other class negatively oriented.
- Standard orientation on $\mathbb{R}^n$ is determined by the standard basis $\{e_1, \dots, e_n\}$. A basis $\{a_1, \dots, a_n\}$ is positively oriented w.r.t. to the standard orientation, if $\det((a_1, \dots, a_n)) > 0$.
- Given two n-dim. oriented vector spaces V and W, then a linear isomorphism $A: V \to W$ is orientation preserving,

if A maps a (hence any) positively oriented basis to a positively oriented basis. Otherwise, it is called orientation-reversing.

On manifolds we can talk about orientations on the tangent spaces; we need notion of smoothness:

Def. 5.1 M mfd.

① M is called orientable, if one can choose an orientation on T_xM $\forall x \in M$ such that the following holds:

For any local frame $\{s_1, \ldots, s_n\}$ on an open connected subset $U \subseteq M$, the basis $\{s_1(y), \ldots, s_n(y)\}$ of $T_yU = T_yM$ is either positively oriented

$\forall y \in U$ or negatively oriented $\forall y \in U$.

② If M is orientable, a choice of orientation on $T_x M$ $\forall x \in M$ as in ① is called an orientation on M.

An orientable mfd. with a chosen orientation is called an oriented manifold.

- If M is connected and orientable, it is easy to see that an orientation is already by the choice of an orientation on one tangent space.

Hence, on a connected orientable mfd. $\exists$ exactly two orientations.

- An open subset U of an oriented mfd. M is itself in a natural way oriented mfd.

Def. 5.2 Suppose M and N are oriented mfds. and $f: M \to N$ a local diffeom. Then f is called orientation preserving, if the lin. iso. $T_x f : T_x M \to T_{f(x)} N$ is orientation-preserv. $\forall x \in M$.

Orientation via special atlases:

Def. 5.3 M mfd.

① An oriented atlas on M is an atlas $\mathcal{A} = \{(U_\alpha, u_\alpha) ; \alpha \in I\}$ for M s.t. for any $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$ the transition map

$u_\beta \circ u_\alpha^{-1} : u_\alpha(U_{\alpha\beta}) \to u_\beta(U_{\alpha\beta})$ has the property that

$\det(D(u_\beta \circ u_\alpha^{-1})) > 0$ on $u_\alpha(U_{\alpha\beta})$.

② Two oriented atlases on M are called orientation equivalent, if their union is again an oriented atlas.

Prop. 5.4 M mfd. of dim. n. Then the following are equivalent:

① M is orientable

② M admits an oriented atlas

③ $\exists$ n-form $\omega \in \Omega^n(M)$ s.t. $\omega(x) \neq 0$ $\forall x \in M$.

Proof.

① $\Rightarrow$ ② Suppose M is orientable and fix an orientation.
Choose atlas on M, $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ such that U_α is connected $\forall \alpha \in I$.

$$\Rightarrow \quad T_x u_\alpha : T_x U_\alpha \xrightarrow{\sim} \underset{\subseteq \mathbb{R}^n}{T_{u_\alpha(x)} u_\alpha(U_\alpha)}$$ is either orientation preserving $\forall x \in U_\alpha$ or orient.-reversing $\forall x \in U_\alpha$.

In the first case we keep the chart as it is, in the second case we compose the chart with an orientation-reversing linear map $\mathbb{R}^n \to \mathbb{R}^n$ (exchange the first two variables).

In this way we obtain an oriented atlas.

② $\to$ ③ Let $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ be an oriented atlas for M. and $\{f_i\}_{i \in \mathbb{N}}$ a partition of unity that is subordinate to $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$.

For each $i \in \mathbb{N}$ choose $\alpha_i \in I$ s.t. $\mathrm{supp}(f_i) \subset U_{\alpha_i}$ and define $\omega^i \in \Omega^n(M)$ by $\omega^i := f_i \, du^1_{\alpha_i} \wedge \dots \wedge du^n_{\alpha_i}$ (extended by zero to M via bump fct.).

Set $\omega := \sum_{i \in \mathbb{N}} \omega^i$ Since $\mathrm{supp}(f_i)$ is locally finite, ω is a smooth n-form on M.

Fix $x \in M$. Then $\sum_{i \in \mathbb{N}} f_i(x) = 1$ implies $\exists\, i$ s.t. $f_i(x) > 0$.

By definition, $\underbrace{\omega^i(x)\left(\frac{\partial}{\partial u^1_{\alpha_i}}, \dots, \frac{\partial}{\partial u^n_{\alpha_i}}\right)}_{= f_i(x)} > 0$.

Since the atlas is oriented and all f_j's have non-negative values on M, we have $\omega^j(x)\left(\frac{\partial}{\partial u^1_{\alpha_i}}, \dots, \frac{\partial}{\partial u^n_{\alpha_n}}\right) \geq 0 \;\forall j$.

$\Longrightarrow \omega(x) \neq 0$.

3 $\longrightarrow$ 1 $\quad \omega \in \Omega^n(M)$ nowhere vanishing.

For $x \in M$ we call a basis $\{s_1, \dots, s_n\}$ of T_xM positively oriented, if $\omega(x)(s_1, \dots, s_n) > 0$.

This yields orientation on M.

$\square$

Remark

- Every oriented atlas determines an orientation on M and two oriented atlases on M determine the same orientation if they are orientation-equivalent.

Similarly, any nowhere vanishing n-form ω determines an orientation and any other such form τ determines the same orientation if $\exists$ positive C^∞-map $f: M \to \mathbb{R}$ s.t. $\omega = f\tau$.

- Given an orientable mfd M, a choice of orientation on M

is equivalent to a choice of maximal oriented atlases (an equiv. class of oriented atlases) and also to a choice of nowhere vanishing n-form up to multiplication by a positive smooth fct.

Ex. $M = \mathbb{R}^n$ is orientable.

Ex. Möbius band is not orientable.

Ex. $\mathbb{RP}^n$ is orientable $\Longleftrightarrow$ n is odd.

5.2 Integrals

Suppose M is an oriented n-dim. mfd. and let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in J\}$ be an oriented atlas representing the orientation.

We write $\operatorname{supp}(\omega) := \overline{\{x \in M : \omega(x) \neq 0\}}$ for the support of a diff. form $\omega \in \Omega^k(M)$ and $\Omega^k_c(M)$ for the space of k-forms with compact support.

Suppose $\omega \in \Omega^n_c(M)$. Since $\operatorname{supp}(\omega)$ is compact, $\exists$ finitely many charts (U_i, u_i) $i = 1, \dots, \ell$ of the oriented atlas $\mathcal{A}$ s.t. $\operatorname{supp}(\omega) \subset U_1 \cup \dots \cup U_\ell$.

Further, let $f_j : M \to [0,1]$ be smooth fcts. $j = 1, \dots, \ell$ s.t. $\operatorname{supp}(f_j) \subset U_j$ and $\displaystyle \sum_{j=1}^{\ell} f_j \Big|_{\operatorname{supp}(\omega)} = 1$.

(Choose a partition of unity subordinate to $\{U_1, \dots, U_\ell, M \setminus \operatorname{supp}(\omega)\}$ and let f_1 be the sum of all fcts with support in U_1, f_2 the sum of the remaining fcts. with support in U_2 and so on ...),

$$\int_M \omega := \sum_{i=1}^{\ell} \int_{u_i(U_i)} \overbrace{(f_i \omega)}^{u_i^*(f_i\omega)(e_1 \dots, e_n)}\underbrace{(u_i^{-1}(y))\left(T_y u_i^{-1} e_1, \dots, T_y u_i^{-1} e_n\right)}_{f_i\omega(u_i^{-1}(y))\left(\frac{\partial}{\partial u_i^1}(u_i^{-1}(y)) \dots, \frac{\partial}{\partial u_i^n}(u_i^{-1}(y))\right)}$$

$\omega = \sum_{i=1}^{\ell} f_i \omega \quad f_i \omega \in \Omega_c^n(U_i)$.

$\mathrm{Supp}(f_i\omega)$ is a compact set of U_i, so the right-hand-side equals a finite sum of integrals of fcts. with compact supports, and hence this integral is finite.

Let us check that $\int_M \omega$ is well-defined, i.e. independent of all choices we made: Suppose (V_j, ψ_j) are finitely many charts of the maximal atlas determined by $\mathcal{U}$ and $g_j : M \to [0,1]$ finitely many smooth fcts as above cover. with respect to these charts.

$\Rightarrow$ $\omega = \sum_j g_j \omega$ and hence $\underline{f_i \omega = \sum_j f_i g_j \omega}$

$$\Rightarrow \sum_i \int_{u_i(U_i)} f_i \omega(u_i^{-1}(y))\,(\underline{T_y u_i^{-1} e_1, \ldots, T_y u_i^{-1} e_n})$$

$$= \sum_{i,j} \int_{u_i(U_i)} f_i g_j \omega(u_i^{-1}(y))\,(\text{—}, \text{———})$$

$$= \sum_{\substack{i,j \\ \operatorname{supp}(f_i g_j \omega) \\ \subset U_i \cap V_j}} \underbrace{\int_{u_i(U_i \cap V_j)} f_i g_j \omega(\text{—}, \text{—})\,(\text{—}, \text{———})}$$

$$\int_{u_i(U_i \cap V_j)} f_i g_j \omega(u_i^{-1}(y))\,(T_u u_i^{-1} e_1, \ldots, T_y u_i^{-1} e_n) =$$

$$= \int_{u_i(U_i \cap V_j)} \det(D(v_j \circ u_i^{-1})) f_i g_j \omega(u_i^{-1}(y))\,(T_{v_j \circ u_i^{-1}(y)} v_j^{-1} e_1, \ldots, T_{v_j \circ u_i^{-1}(y)} v_j^{-1} e_n)$$

$$= \int_{\underline{v_j(U_i \cap V_j)}} f_i\, g_j\, \omega\, (v_j^{-1}(y))\,(T v_j^{-1} e_1, \dots, T v_j^{-1} e_n) =$$

$$= \int_{v_j(V_j)} \text{———} \;''\; \text{——————}$$

since $f_i g_j \omega$ vanishes outside $v_j(U_i \cap V_j)$.

Hence, $\int_M \omega$ is well-defined.

<u>Prop. 5.5</u> M mfd. of dim n.

Then $\int_M : \Omega^n_c(M) \longrightarrow \mathbb{R}$ is a surjective linear map.

Proof. Linearity follows from the definition and linearity of integral of fct. on $\mathbb{R}^n$.

Surjectivity: It remains to show that $\exists\, \omega \in \Omega^n_c(M)$ s.t.

$$\int_M \omega \neq 0 .$$

Choose a chart (U, u) and a smooth non-zero fct. $f: M \to \mathbb{R}_{\geq 0}$ with compact support in U. $\omega := f\, du^1 \wedge \dots \wedge du^n$ can be extended by zero to an element in $\Omega^n_c(M)$.

$$\Rightarrow \int_M \omega = \int_{u(U)} f \circ u^{-1} > 0 .$$

Special cases

① $M = \mathbb{R}$ equipped with standard orientation, for $a<b$
and $\omega = f\,dt \in \Omega^1(\mathbb{R})$

$$\int_{[a,b]} \omega = \int_a^b f(t)\,dt .$$

② Line integrals: $V \subseteq \mathbb{R}^n$ open subset, $\omega = \sum_{i=1}^{n} \omega_i\, dx^i \in \Omega^1(V)$.
and $\gamma : I \to V$ C^∞-curve, $I \subseteq \mathbb{R}$ open interval.

$$\rightarrow \int_I \gamma^* \omega = \sum_{i=1}^{n} \int (\omega_i \circ \gamma)(\gamma_i'(t))\,dt \qquad \left(\int_\gamma \omega\right).$$

line integral of ω along γ.