Tutorial 3

Homework 3

$f, g \in C^0(N)$ $\quad (f+g)(n) := f(n) + g(n)$ $\mathbb{R}$.

(1) M, N topolog. spaces, $F : M \to N$ continuous map.

$F^* : C^0(N) \longrightarrow C^0(M)$ $\qquad F^*(f) = f \circ F : M \to \mathbb{R}$

(a) F^* linear,

$f, g \in C^0(N)$, $\lambda \in \mathbb{R}$ $\quad \Longrightarrow \boxed{F^*(f + \lambda g) = F^* f + \lambda F^* g.}$

$F^*(f + \lambda g)(m) = ((f + \lambda g) \circ F)(m) \overset{\checkmark}{=} \underline{f(F(m))} + \lambda g(F(m))$

$(f + \lambda g)(F(m))$

$= (F^* f + \lambda F^* g)(m).$

(b) Assume M, N (smooth) mfds.

F is smooth $\iff$ $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

'$\Rightarrow$' $g \in C^\infty(N)$

$F^* g = g \circ F : M \to \mathbb{R}$ is smooth as composition of smooth fcts.

'$\Leftarrow$' Fix $x \in M$ and let (U, u) be a chart of N with $F(x) \in U$. We can extend u to a smooth fct

$\tilde{u} : N \to \mathbb{R}^n$ $\tilde{u} = \begin{pmatrix} \tilde{u}^1 \\ \vdots \\ \tilde{u}^n \end{pmatrix}$

s.t. (w.l.o.g.) $\tilde{u}|_U = u$.

By assumption, $F^*\tilde{u}^i = \tilde{u}^i \circ F : M \to \mathbb{R}$ is smooth $\forall i$.

Suppose (V, v) is a chart on N with $x \in V$.

$$\Rightarrow \quad \underline{\tilde{u} \circ F} \circ v^{-1}\big|_{v(V \cap F^{-1}(U))} : v(V \cap F^{-1}(U)) \longrightarrow \tilde{u}(U) = u(U)$$

$$\| $$

$$\underline{u \circ F \circ \underline{v}^{-1}}$$

is smooth as comp. of smooth ~~fct~~ maps.

Hence, F is smooth by def. of smoothness for maps $M \to N$.

(c) Assume F is a homeom. between smooth mf M and N.

$F^{-1} : N \longrightarrow M$ continuous inverse.

F is a diffeom. $\Longleftrightarrow$ $F^* : C^\infty(N) \to C^\infty(M)$ is

an isomorphism.

$F \circ F^{-1} = Id_N \qquad F^{-1} \circ F = Id_M$

$$\begin{array}{c} (F \circ F^{-1})^* = Id_N^* = Id_{C^\infty(N)} \\ \| \\ (F^{-1})^* \circ F^* \end{array} \qquad (F \circ F^{-1})^* g = g \circ F \circ F^{-1} = (F^{-1})^* F^* g$$

$\Longrightarrow \boxed{(F^{-1})^* = (F^*)^{-1}}$

So the statement follows from (b).

$u = \mathrm{id}$

(2) $\xi(x,y,z) = \underline{2}\,\frac{\partial}{\partial x} \underline{-} \frac{\partial}{\partial y} \underline{+3}\,\frac{\partial}{\partial z}$ v.f. on $\mathbb{R}^3$

In cylindrical coordinates: $v^{-1}(r,\varphi,z) = (r\cos\varphi, r\sin\varphi, z=z)$

Jacobian of $u \circ v^{-1} = \mathrm{id} \circ v^{-1}$

$$\begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse is

$$\frac{1}{r}\begin{pmatrix} r\cos\varphi & r\sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & r \end{pmatrix}$$

$$\frac{1}{r}\begin{pmatrix} r\cos\varphi & r\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & r \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2\cos\varphi - \sin\varphi \\ -\frac{2}{r}\sin\varphi - \frac{1}{r}\cos\varphi \\ 3 \end{pmatrix}$$

$$\xi = (2\cos\varphi - \sin\varphi)\frac{\partial}{\partial r} - \frac{2\sin\varphi + \cos\varphi}{r}\frac{\partial}{\partial \phi} + 3\frac{\partial}{\partial z}.$$

In spherical coordinates:

$$\nu^{-1}(r, \varphi, \theta) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$$

Jacobian of ~~[illegible]~~: void

$$-\frac{1}{r^2\sin\theta}\begin{pmatrix} -r^2\sin^2\theta\cos\varphi & -r^2\sin^2\theta\sin\varphi & -r^2\sin\theta\cos\theta \\ r\sin\varphi\,\cancel{\theta} & -r\cos\varphi & 0 \\ -r\sin\theta\cos\theta\cos\varphi & -r\sin\theta\cos\theta\sin\varphi & r\sin^2\theta \end{pmatrix}$$

$$\Rightarrow \zeta = \left(2\cos\varphi\sin\theta - \sin\theta\sin\varphi + 3\cos\theta\right)\frac{\partial}{\partial r}$$

$$-\frac{2\sin\varphi + \cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$+\frac{2\cos\theta\cos\varphi - \cos\theta\sin\varphi - 3\sin\theta}{r}\frac{\partial}{\partial\theta}$$

Jacobian $\mathrm{id} \circ v^{-1}$:

$$\begin{pmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta \end{pmatrix}$$

(3) $M = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$

$M = f^{-1}(0) \qquad f(x,y,z) = x^2+y^2-1$

$T_{(x,y,z)}M = \ker\left(T_{(x,y,z)} f\right) = \{(x',y',z') \in \mathbb{R}^3 ; xx'+yy' = 0\}$

$s \cdot f(x,y,z) = T_x f \begin{pmatrix} x^2-1 \\ xy \\ xz \end{pmatrix} = 2x\left(\underline{x^2+y^2}-1\right) = 0$

for $(x,y,z) \in M$.

Hence, ξ is tangent to M (i.e. $\xi|_M \in \mathfrak{X}(M)$).

$$\eta \cdot f(x,y,z) = T_x f \begin{pmatrix} x \\ y \\ 2xz^2 \end{pmatrix} = 2x^2 + 2y^2 \neq 0$$

for $(x,y,z) \in M$.

$\Longrightarrow$ η is not tangent to M.

(4) $M = \mathbb{R}^2$ $\quad \xi(x,y) = y \frac{\partial}{\partial x}(x,y) \quad \eta(x,y) = \frac{x^2}{2} \frac{\partial}{\partial y}(x,y)$.

$$[\xi, \eta] = \left[y \frac{\partial}{\partial x}, \frac{x^2}{2} \frac{\partial}{\partial y} \right] = yx \frac{\partial}{\partial y} - \frac{x^2}{2} \frac{\partial}{\partial x}.$$

$$c(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$ is an integral curve for $[s,n]$

$$\iff \quad c_1'(t) = -\frac{c_1^2(t)}{2}$$

$$c_2'(t) = c_1(t)\, c_2(t).$$

$$\Longrightarrow \quad c_1(t) = \frac{2}{t+a}$$ a constant

$$c_2(t) = (t+a)^2 \cdot b$$ b constant.

Initial condition: $c(0) = (x,y) \Longrightarrow a = \frac{2}{x}$ and $b = \frac{x^2 y}{4}$

$$\Longrightarrow \quad c_{(x,y)}(t) = F^{[s,n]}(t, (x,y)) = \left(\frac{2x}{2+tx},\ \left(t + \frac{2}{x}\right)^2 \frac{x^2 y}{4} \right)$$

Not defined for $t = -\frac{z}{x}$; hence $[s, \eta]$ is not complete.

Some applications of the Frobenius Thm:

<u>Ex.</u> Consider the following linear system of PDEs for $f: \mathbb{R}^3 \to \mathbb{R}$

$$(*) \quad \begin{aligned} -2z^2 \frac{\partial f}{\partial x} + 2x \frac{\partial f}{\partial z} &= 0 \\ -3z^2 \frac{\partial f}{\partial y} + 2y \frac{\partial f}{\partial z} &= 0 \end{aligned}$$

Does $(*)$ has any non-constant solutions ?

$$\left.\begin{aligned} X &= -2z^2 \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial z} \\ Y &= -3z^2 \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial z} \end{aligned}\right\}$$

Span a rank 2 distribution E on the subset

$V = \{ (x,y,z) \in \mathbb{R}^3 : z \neq 0 \}$
$\subseteq \mathbb{R}^3$

$(X \cdot f = 0 \quad Y \cdot f = 0 \quad \text{is } (*))$.

Moreover, $[X,Y] = -12xz \frac{\partial}{\partial y} + 8yz \frac{\partial}{\partial x} = \frac{4x}{z} Y - \frac{4y}{z} X$

Hence, E is integrable by Frob. Thm.

$\Rightarrow \exists$ locally around any $(x_0, y_0, z_0) \in V$ a chart (U, u) s.t. E is spanned by $\frac{\partial}{\partial u^1}$ and $\frac{\partial}{\partial u^2}$.

(*) in coordinates $\begin{pmatrix} u^1(x,y,z) \\ u^2(x,y,z) \\ u^3(x,y,z) \end{pmatrix}$ is equiv. to $\frac{\partial f}{\partial u^1} = \frac{\partial f}{\partial u^2} = 0$.

Hence, $f = u^3$ is a solution and any solution in a suffic. small neighbh. of (x_0, y_0, z_0) is of the form $f(x,y,z) = g(u^3(x,y,z))$ for smooth fct. g in one variable.

<u>Ex.</u> System of PDEs for $f: \mathbb{R}^2 \to \mathbb{R}$.

- $\frac{\partial f}{\partial x}(x,y) = \alpha(x, y, f(x,y))$ (**)
- $\frac{\partial f}{\partial y}(x,y) = \beta(x, y, f(x,y))$

α, β smooth fcts def. on open subset $V \subseteq \mathbb{R}^3$.

Q. When does (**) has a solution?

Necessary condition for α and β:

$$\boxed{\frac{\partial}{\partial y}\alpha\left(x,y,f(x,y)\right) = \frac{\partial}{\partial x}\beta\left(x,y,f(x,y)\right)}$$

since $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

Chain rule $\Rightarrow$ $$\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z} \quad (***).$$

$(***)$ is necess. condition for $(**)$ to have a solution in a neighbh. of any point (x_0, y_0) with arbitrary value $f(x_0, y_0) = z_0$.

By Frob. Thm. $(***)$ is also sufficient: it implies that for any $(x_0, y_0, z_0) \in V$ $\exists$ an open neigh. U of $(x_0, y_0) \in \mathbb{R}^2$ and a unique solution $f: U \to \mathbb{R}$ of $(**)$ with $f(x_0, y_0) = z_0$.

<u>Why?</u> $(**)$ prescribes the tangent plane to the graph of f. Collection of the tangent planes defines rank 2 distribution of V and $(***)$ is equiv. to involutivity.

Suppose $f : U \to \mathbb{R}$ were a solution ($U \subseteq \mathbb{R}^2$ open subset) of (*x)

Then $\psi : U \to \mathbb{R}^3$

$$\psi(x,y) = (x, y, f(x,y))$$ (ψ param. of surf. $gr(f)$.)

is a diffeom. onto $gr(f)$.

$T_{\psi(x,y)} gr(f)$ is spanned by

$$T_{(x,y)}\psi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial z} \quad \parallel \alpha(x,y,f(x,y))$$

$$T_{(x,y)}\psi\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial z} \quad \beta(x,y,f(x,y)).$$

$$X = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial z}$$

v.f. on V.

$$Y = \frac{\partial}{\partial y} + \beta \frac{\partial}{\partial z}$$

Span a rank 2 distr. E on V.

E is involutive $\iff$ (***) holds.

& In this case through any point $(x_0, y_0, z_0) \in V$

$\exists$ an integral submfd. $N \subseteq V \subseteq \mathbb{R}^3$ which locally has

the form $gr(f)$ for a smooth fct. $f: U \to \mathbb{R}$, U open

$f(x_0, y_0) = z_0$. nbhd. of (x_0, y_0)