Tutorial 4

① M mfd., $\xi, \eta \in \mathfrak{X}(M)$.

ⓐ $\overset{(i)}{[\xi,\eta] = 0} \iff \overset{(ii)}{Fl_t^{\xi\,*}\eta = \eta}$ (whenever defined)

$\iff Fl_t^{\xi} \circ Fl_s^{\eta} = Fl_s^{\eta} \circ Fl_t^{\xi}$ (whenever defined). (iii)

Recall : $[\xi,\eta](x) = \frac{d}{dt}\Big|_{t=0} (Fl_t^{\xi})^* \eta(x) \quad \forall x \in M.$

(ii) $\Rightarrow$ (i)

$t \mapsto (Fl_t^{\xi})^* \eta(x) = \eta(x)$ is a constant curve defined on interval around 0

$\Rightarrow \quad 0 = \frac{d}{dt}\Big|_{t=0} (Fl_t^{\xi})^* \eta(x) = [\xi,\eta](x)$

(i) $\Longrightarrow$ (ii)

$$\underline{(Fl^{\xi}_t)^* \eta (x)} = \big(\underbrace{(TFl^{\xi}_t)^{-1}}_{TFl^{\xi}_{-t}} \eta (Fl^{\xi}_t(x))\big)$$

Thm. 3.19

$$\frac{d}{dt} \underline{(Fl^{\xi *}_t \eta)(x)} = \frac{d}{ds}\Big|_{s=0} (Fl^{\xi}_{t+s})^* \eta(x) = \frac{d}{ds}\Big|_{s=0} (Fl^{\xi}_s \circ Fl^{\xi}_{t})^* \eta(x)$$

$$= \frac{d}{ds}\Big|_{s=0} (Fl^{\xi}_t)^* (Fl^{\xi}_s)^* \eta(x) = (Fl^{\xi}_t)^* \underline{\frac{d}{ds}\Big|_{s=0} (Fl^{\xi}_s)^* \eta(x)}$$

$$= (Fl^{\xi}_t)^* [\xi, \eta](x) = 0$$

$\Longrightarrow$ $t \mapsto Fl^{\xi *}_t \eta(x)$ is constant and since $(Fl^{\xi}_0)^* \eta(x) = \eta(x)$, we have $(Fl^{\xi}_t)^* \eta(x) = \eta(x)$ whenever defined.

ii $\Longleftrightarrow$ iii

Thm 3.19

$$Fl^{\xi}_t \circ Fl^{\eta}_s = Fl^{\eta}_s \circ Fl^{\xi}_t \iff Fl^{\eta}_s = Fl^{\xi}_{-t} \circ Fl^{\eta}_s \circ Fl^{\xi}_t$$

$$\overset{(*)}{=} Fl_s^{Fl^{\xi *}_t \eta}$$

,

$$\iff \eta = Fl^{\xi *}_t \eta$$

$\wedge$

Why $(*)$? :

$$\frac{d}{ds}\Big|_{s=0} \left(Fl^{\xi}_{-t} \circ Fl^{\eta}_s \circ Fl^{\xi}_t\right)(x) = T_{Fl^{\xi}_t(x)} Fl^{\xi}_{-t}\, \eta\left(Fl^{\xi}_t(x)\right)$$

$$= \left(Fl^{\xi}_t\right)^* \eta(x) .$$

Remark to (*):

More generally, suppose $f: M \to N$ C^∞-map between mfds and $\xi \in \mathfrak{X}(M)$, $\eta \in \mathfrak{X}(M)$ are f-related (i.e. $Tf\,\xi = \eta \circ f$).

$$\Longrightarrow \boxed{f \circ Fl_t^{\xi} = Fl_t^{\eta} \circ f}$$

(in particular, if f is a diffeom., then $Fl_t^{\xi} = f^{-1} \circ Fl_t^{\eta} \circ f$)

Proof:

$$\frac{d}{dt}\left((f \circ Fl_t^{\xi})(x)\right) = T_{Fl_t^{\xi}(x)} f \; \xi\left(Fl_t^{\xi}(x)\right) =$$

$$= \eta\left(f\left(Fl_t^{\xi}(x)\right)\right)$$

$\Longrightarrow$ $t \mapsto f(Fl_t^{\xi}(x))$ is an integral curve of η through $f(x)$.

$$\Longrightarrow f\left(Fl_t^{\xi}(x)\right) = (Fl_t^{\eta} \circ f)(x).$$

(b) $f: M \to N$ C^∞-map, $\xi \in \mathfrak{X}(M)$ f-rel. to $\tilde{\xi} \in \mathfrak{X}(N)$
$\eta \in \mathfrak{X}(N)$ f-rel. to $\tilde{\eta} \in \mathfrak{X}(N)$.

Then $[\xi, \eta]$ is then f-rel. to $[\tilde{\xi}, \tilde{\eta}]$.

<u>Proof.</u> $h \in C^\infty(N, \mathbb{R})$, $h \circ f \in C^\infty(M, \mathbb{R})$.

$$\underline{\xi \cdot (h \circ f)(x)} = \xi_x \cdot (h \circ f) = (T_x f\, \xi_x) \cdot h = \tilde{\xi}_{f(x)} \cdot h = \underline{((\tilde{\xi} \cdot h) \circ f)(x)}$$

and the same of course for η and $\tilde{\eta}$.

$$\Rightarrow \underline{[\xi, \eta] \cdot (h \circ f)} = \xi \cdot (\underline{\eta \cdot (h \circ f)}) - \eta \cdot (\underline{\xi \cdot h \circ f})$$

$$= \underline{\xi \cdot ((\tilde{\eta} \cdot h) \circ f)} - \eta \cdot ((\tilde{\xi} \cdot h) \circ f) = \underline{\tilde{\xi} \cdot (\tilde{\eta} \cdot h)} \circ f - \underline{\tilde{\eta} \cdot (\tilde{\xi} \cdot h)} \circ f =$$

$= ([\hat{\xi}, \hat{\eta}] \cdot h) \circ f$

$\Rightarrow (Tf\,[\xi, \eta]) \cdot h = [\hat{\xi}, \hat{\eta}] \cdot h \circ f$

$\Rightarrow \underline{Tf[\xi, \eta] = [\hat{\xi}, \hat{\eta}] \circ f}$

(2) $GL(n, \mathbb{R})$ $A \in GL(n, \mathbb{R})$

(a) $\lambda_A : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $\rho_A : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$.

are bijections with inverse $\lambda_{A^{-1}}$ and $\rho_{A^{-1}}$.

Smoothness of these maps follows from smoothness of

$\mu : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$. ($\lambda_A = \mu \circ i_B$

$i_B : B \to (A, B) \in GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$)

Note that λ_A, ρ_A are restrictions to $GL(n,\mathbb{R})$ of linear maps $M_n(\mathbb{R}) \to M_n(\mathbb{R})$.

$$\Rightarrow \quad T_B \lambda_A (B, X) = (AB, AX)$$
$$T_B \rho_A (B, X) = (BA, XA).$$

⑤ Compute the tangent map of $\mu : GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$.

$$T_{(A,B)} (GL(n,\mathbb{R}) \times GL(n,\mathbb{R})) = T_A GL(n,\mathbb{R}) \times T_B GL(n,\mathbb{R})$$

elements in are of the form.

$$((A,B), (X,Y)) \qquad X, Y \in M_n(\mathbb{R}).$$

$c(t) = (c_1(t), c_2(t))$ $\qquad$ $c_1(t) = A + tX$ $\qquad$ $AB + tXB + tAY + t^2XY$

$c_2(t) = B + tY$ $\qquad$ $= (A+tX)(B+tY)$

$$\underline{T_{(A,B)}\mu\left((A,B),(X,Y)\right)} = \left(AB, \frac{d}{dt}\Big|_{t=0} \overbrace{\mu(c_1(t), c_2(t))}^{(A+tX)(B+tY)}\right)$$

$$= (AB, XB + AY)$$

$$= \underline{T_B\lambda_A Y + T_A\rho^B X}$$

(c) $X \in M_n(\mathbb{R})$
$\quad \simeq T_{\mathrm{Id}}GL(n,\mathbb{R})$

$L_X(B) = \underline{(B, BX)}$

$R_X(B) = \overline{(B, XB)}$ $\qquad$ smooth vector fields on $GL(n,\mathbb{R})$

A vector field ξ on $GL(n,\mathbb{R})$ is called left-invariant (resp. right-invariant), if $\lambda_A^* \xi = \xi \quad \forall A \in GL(n,\mathbb{R})$

(resp. if $\rho_A^* \xi = \xi \quad \forall A \in GL(n,\mathbb{R})$)

$$(\lambda_A^* L_X)(B) = (T\lambda_A)^{-1} L_X(AB) = T\lambda_{A^{-1}}(AB, ABX)$$
$$= (B, BX) = L_X(B)$$

$$\rho_A^* R_X(B) = T\rho_{A^{-1}} R_X(BA) = T\rho_{A^{-1}}(BA, XBA)$$
$$= (B, XB) = R_X(B)$$

Remark In fact any left-inv. vector field (resp. right-inv. v.f.) is of the form L_X for some $X \in M_n(\mathbb{R})$ (resp. R_X for $X \in M_n(\mathbb{R})$)

Suppose $\xi \in \mathcal{X}(GL(n,\mathbb{R}))$ is left-invariant.

$$(\lambda_A^* \xi)(B) = \underline{\xi(B)} \qquad \forall A, B \in GL(n,\mathbb{R}).$$

$$\| $$

$$\underline{(T\lambda_A)^{-1} \xi(AB)}$$

$$\xi(AB) = T_B \lambda_A \xi(B)$$

In particular, $\xi(A) = T_{Id} \lambda_A \xi(Id)$

$$\xi(Id) = (Id, X) \qquad = L_{\xi(Id)}(A)$$

$$\simeq X.$$

Similarly, for right invariant vector field.

Flows?

$c : I \to GL(n,\mathbb{R})$ is an integral curve of L_X through $B \in GL(n,\mathbb{R})$,

L_X

if $c'(t) = L_X(c(t)) = c(t)X$

$c(0) = B$

$\frac{d}{dt} Be^{tX} = Be^{tX}X = c(t)X$

$\Longrightarrow c(t) = Be^{tX}$.

$e^{tX} = \sum \frac{t^k X^k}{k!}$

For R_X: $c'(t) = R_X(c(t)) = X c(t)$

$c(0) = B$

$\Longrightarrow c(t) = e^{tX} B$ integral curve of R_X through B.

Integral curves defined $\forall t \Longrightarrow L_X$ and R_X are complete.

d) $[L_X, R_Y] = 0 \overset{1(a)}{\Longleftrightarrow}$ their flows commute.

$$Fl^{L_X}_t \circ Fl^{R_Y}_s (B) = Fl^{L_X}_t (e^{sY} B) = e^{sY} B e^{tX}$$
$$= Fl^{R_Y}_s (Fl^{L_X}_t (B))$$

Remark Note that statements (a) - (d) are valid for any matrix group $G \subseteq GL(n,\mathbb{R})$ (i.e. Lie subgr. of $GL(n,\mathbb{R})$).

- $A \in G \leadsto \lambda_A \quad \rho_A$
- L_X, R_X for any $X \simeq (Id, X) \in T_{Id} G$.

$[X,Y] = L_{[X,Y]}(Id)$

($T_{Id} GL(n,\mathbb{R}) \quad X \longmapsto L_X \quad L_{[X,Y]} = [L_X, L_Y]$

$[\,,\,]$ commutator is Lie bracket.

$= [L_X, L_Y](Id)$.

③ Consider n vector fields on $\mathbb{R}^{n+k}$ $\quad (x,z) = (x^1,\dots,x^n,z^1,\dots,z^k)$.

$$X_j = \frac{\partial}{\partial x^j} + \alpha_j^\ell(x,z)\frac{\partial}{\partial z^\ell}$$

(Einstein summation convention; 2nd term equals $\sum_{\ell=1}^{k} \alpha_j^\ell(x,z)\frac{\partial}{\partial z^\ell}$).

$j=1,\dots,n$

These vector fields pairwise commute:

$$[X_j, X_m] = \left[\frac{\partial}{\partial x^j} + \alpha_j^\ell \frac{\partial}{\partial z^\ell}, \frac{\partial}{\partial x^m} + \alpha_m^r \frac{\partial}{\partial z^r}\right]$$

$$= \frac{\partial \alpha_m^r}{\partial x^j}\frac{\partial}{\partial z^r} - \frac{\partial \alpha_\ell^\ell}{\partial x^m}\frac{\partial}{\partial z^\ell} + \alpha_j^\ell \frac{\partial \alpha_m^r}{\partial z^\ell}\frac{\partial}{\partial z^r} - \alpha_m^r \frac{\partial \alpha_j^\ell}{\partial z^r}\frac{\partial}{\partial z^\ell}$$

$$= \overbrace{\left(\frac{\partial \alpha_m^\ell}{\partial x^j} + \alpha_j^r \frac{\partial \alpha_m^\ell}{\partial z^r} - \frac{\partial \alpha_j^\ell}{\partial x^m} - \alpha_m^r \frac{\partial \alpha_j^\ell}{\partial z^r}\right)}^{=0 \text{ by assumption.}} \frac{\partial}{\partial z^\ell}.$$

Hence. $[X_j, X_m] = 0$

$\Rightarrow$ $\exists$ a coordinate chart $\tilde{u} : \tilde{U} \to \tilde{u}(U) = \tilde{W} \times \tilde{W} \subseteq \mathbb{R}^n \times \mathbb{R}^k$

locally around any point $(x_0, z_0) \in U$ s.t.

$$X_j\Big|_{\tilde{U}} = \frac{\partial}{\partial \tilde{u}^j} \qquad j = 1, \dots, n.$$

For each $a \in \tilde{W}$, $u^{-1}(W \times \{a\})$ integral mf. of $(*)$

$$\left\langle \frac{\partial}{\partial \tilde{u}^1}, \dots, \frac{\partial}{\partial \tilde{u}^n} \right\rangle = \left\langle X_1\Big|_{\tilde{U}}, \dots, X_n\Big|_{\tilde{U}} \right\rangle.$$

$u_{n+1} = \dots = u_{n+k} = a$ is the equation for $(*)$.

Implicit fct. Thm $\Rightarrow$ integral subsets - ~~to~~ in a neighbh. V of x_0 in $\mathbb{R}^n$ can be written

as the graph of a fct. $f : V \to \mathbb{R}^k$ with $f(x_0) = z_0$.

Tangent space of $gr(f)$ ~~at~~ is given by

$$T_{\psi(x)}\, gr(f) = \operatorname{Im}(T_x\psi) \quad (*)$$

$\psi(x) = (x, f(x))$. $\psi : V \to \mathbb{R}^{n+k}$

$(*)$ is spanned by $T_x\psi\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j} + \frac{\partial f^\ell}{\partial x^j}\frac{\partial}{\partial z^\ell}$

for $j = 1, \dots, n$

$$= \underline{\frac{\partial}{\partial x^j} + \alpha_j^\ell \frac{\partial}{\partial z^\ell}}.$$

(4)

(a) (b) (c)

Yes No Yes

$$\frac{\partial^2 f}{\partial y \partial x} = -f \sin y + \frac{\partial f}{\partial y} \cos y$$

$$= -f \sin y - f \log f \tan y \cos y$$

$$\frac{\partial f}{\partial x} = \underbrace{f \cos y}_{\alpha(x,y,f)} \Rightarrow \frac{\partial f}{\partial y} = \underbrace{-f \log f \tan y}_{\beta(x,y,f)}$$

$$\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial f} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial f}$$

$$\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial z} = -f \sin y - f \log f \underbrace{\tan y \cos y}_{\sin y}$$

$$= -f \sin y \,(1 + \log f) .$$

$$\frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z} = -f \cos y \tan y \,(1 + \log f)$$

$$= -f \sin y \,(1 + \log f)$$