Lecture 11 - Modules

Def) Let R be a ring. A left R-module $(M,+,0)$ is an <u>abelian group</u> M with a function
$$\cdot : R \times M \longrightarrow M$$
satisfying

1. $r.(a+b) = r.a + r.b$
2. $(r+s).a = r.a + s.a$

$\Big\}$ bilinear

3. $(r.s).a = r.(s.a)$
4. $1.a = a$

$\Big\}$ assoc. unital action

- A homomorphism $f: M \longrightarrow N$ of R-modules is a homomorphism of abelian groups such that $f(r.a) = r.f(a)$.

- There is a category <u>Mod_R</u> of left R-modules which has forgetful functors

$$\begin{tikzcd} Mod_R \arrow[r] \arrow[dr] & Ab \arrow[d] \\ & Set \end{tikzcd}$$

Examples

① When $R=K$ is a field, a K-module is a <u>vector space over K</u>.

② When $K=\mathbb{Z}$, a $\mathbb{Z}$-module is exactly an abelian group.

Indeed, if M is an ab. grp., we are forced to define $\cdot : \mathbb{Z} \times M \longrightarrow M$ as follows:

since bilinearity implies each

$$- \cdot m : \mathbb{Z} \longrightarrow M \text{ is homomorph.}$$

of abelian groups, and since $\mathbb{Z}$ is the free ab. group gen. by element 1, we must def.

$$n.m = \underbrace{(1+\ldots+1)}_{n \text{ times}}.m = \underbrace{m+\ldots+m}_{n \text{ times}}$$

& sim. for negative n.

Exercise: check remaining details.

(3) - Let G be a group & R a ring

Then we can

form the group ring $R[G] =$

$\{\lambda_1 g_1 + \ldots \lambda_1 g_k : \lambda_1, \ldots, \lambda_k, g_1, \ldots, g_k \in G\}$

the set of formal R-linear combinations of elements of G.

of elements of G.

- This is an abelian group with obvious addition and has multiplication
$$\left(\sum_{i=1}^{k} \lambda_i g_i\right)\left(\sum_{j=1}^{\ell} \mu_j g_j\right) = \sum_{i=1}^{k}\sum_{j=1}^{\ell} (\lambda_i \mu_j)(g_i g_j)$$
obtained by extending the mult. of G bilinearly.

$R[G]$-modules are often called <u>group</u> <u>representations</u>:

they amount to R-module M with
$$G \times M \xrightarrow{\cdot} M \quad \text{sat}$$
$$e.m = m \quad \&$$
$$(g.h).m = g.(h.m).$$
$\Big\}$ Ex!

This is of most interest when $R = K$ is a field and the vect. space M is fin. dim., so $M = K^n$; then $K[G]$ module:

group hom $G \longrightarrow Gl(n, K)$.

Remark :- $Mod_R = (\Omega, E)$-Alg where

$\Omega = \{ \underbrace{r \cdot -}_{\text{unary operation}} : r \in R, \underbrace{+}_{\text{bin}}, \underbrace{0}_{\text{nullary}} \}$

with

E the equations of the def above.

- From the previous chapter, it follows that Mod_R is complete & cocomplete, that forg. functors $U: Mod_R \longrightarrow Ab, Set$ have left adjoints.

Kernels & quotients

- As for abelian groups, the notions of kernels & quotients for modules are very simple.

Def^n
- Given a homomorphism $f: M \to N$ of modules, its <u>kernel</u> $\ker(f) \hookrightarrow M$ is the submodule $\ker f = \{ x \in M : fx = 0 \}$
- <u>Given a</u> submodule $N \subseteq M$, the <u>quotient M/N</u> $= \{ m+N : m \in M \}$ is the quotient group (set of cosets) with $r.(m+N) = rm+N$. We then have $M \xrightarrow{p^N} M/N$ module homomorphism.

Exercise: Express $\ker f$ as an equaliser in $\mathrm{Mod}\,R$, & M/N as a coequaliser.

- The first isomorphism theorem in this setting says:

Theorem:
- If $f: M \longrightarrow N \in \mathrm{Mod}\,R$, then

$\mathrm{im} f \cong M/\ker f$. In particular, if f is surj.,
then $N \cong M/\ker f$.

- If $N \subseteq M$ a submodule, then
$N = \ker(M \longrightarrow M/N)$.

- This is all routine. What is interesting about $\mathrm{Mod}\,R$ is what happens for products and coproducts.

Products & coproducts

Proposition

- $0 = \{0\}$ is both the terminal & initial object in R-Mod.

Proof

Clearly terminal. 0 is initial as must define $f : 0 \longrightarrow M$ by $f(0) = 0$.

Remark: The zero homomorphism $0 : M \to N$ is the composite $M \longrightarrow 0 \longrightarrow N$. $\square$

- Given a set $(M_i)_{i \in I}$ of R-modules, as for any algebraic category, the product $\prod_{i \in I} M_i \xrightarrow{p_i} M_i$ consists of sequences $\{(a_i)_{i \in I} : a_i \in M_i\}$ with component-wise module structure.

Definition

The direct sum $\bigoplus_{i \in I} M_i \hookrightarrow \prod_{i \in I} M_i$ is the submodule consisting of those $(a_i)_{i \in I}$ for which $a_i \neq 0$ only for finitely many $i \in I$.

Remark :- In particular, if I is a finite set, then $\bigoplus_{i \in I} M_i = \prod_{i \in I} M_i$.

- So for instance,

$$A \oplus B = A \times B.$$

• Observe that there are R-module homomorphisms

$$M_i \xrightarrow{q_i} \bigoplus_{i \in I} M_i$$
$$a \longmapsto q_i(a) \text{ seq. w' } (q_i(a))_i = a, \ (q_i(a))_j = 0 \text{ otherwise.}$$

Theorem The maps $M_i \xrightarrow{q_i} \bigoplus_{i \in I} M_i$ exhibit $\bigoplus_{i \in I} M_i$ as the coproduct.

<u>Proof</u>

- Consider $(f_i : M_i \longrightarrow N)_{i \in I}$ homomorphisms in $\mathrm{Mod}R$.

- We must show that $\exists!$ hom f making

$$\begin{tikzcd} & \bigoplus_{i\in I} M_i \arrow[dr, "f"] & \\ M_i \arrow[ur, "q_i"] \arrow[rr, "f_i"] & & N \end{tikzcd}$$

commute.

- Given $a=(a_i)_{i\in I}\in\bigoplus M_i$, we have $a=q_{i_1}(a_{i_1})+\dots+q_{i_n}(a_{i_n})$ where $i_1,\dots,i_n\in I$ are the values @ which a is non-zero.
- Then for f to be a homomorph. sat. $f\circ q_i=f_i$, we must define
$f(a)=f_{i_1}(a_{i_1})+\dots+f_{i_n}(a_{i_n})$.
- It is straightforward, using that N is commutative, to see that
f is a homomorphism.

• Indeed: given a,b, let
$i_1,\dots,i_n,j_1,\dots,j_m\in I$ be those @ which a,b resp. non-zero.

Then $F(a+b) =$
$f_{i_1}(a_{i_1}+b_{i_1}) + \dots + f_{j_m}(a_{j_m}+b_{j_m})$
$= f_{i_1}a_{i_1} + f_{i_1}b_{i_1} + \dots + f_{j_m}a_{j_m} + f_{j_m}b_{j_m}$
$= \underbrace{f_{i_1}a_{i_1} + \dots + f_{i_n}a_{i_n}} + \underbrace{f_{j_1}a_{j_1} + \dots f_{j_m}b_{j_m}}$
$= Fa + Fb.$ using commut. $\square$

- In particular, $A \oplus B = A \times B$ is both product & coproduct.
- Similarly,
  R^n is both product & coprod. in Mod-R.
- This corresponds to the fact that homomorphisms
  $R^m \xrightarrow{f} R^n$ correspond to matrices: we have bij^{n}s

$$\frac{R^m \xrightarrow{f} R^n}{R \xrightarrow{f_i} R^n \quad i \in \{1, \dots, m\}}$$

as R^m is m-fold coprod

$R \xrightarrow{f_{ij}} R$ $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ as R^n is n-fold product

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Elts $\{A_{ij}\}_{i \in \{1, \dots, m\}, j \in \{1, \dots, n\}}$ of R since a hom. $R \to R$ is specified by value at $1 \in R$ (R is free R-module on 1!)

- Under this correspondence, composition of homomorphisms corresp. to matrix multiplication.

Free R-modules

- Let X be a set. The free R-module FX consists of formal linear combinations
$$\lambda_1 x_1 + \dots + \lambda_n x_n$$
with obvious addition, action.

- In fact, this is equally the direct sum $\bigoplus_{x \in X} R$

Questions :

- What is adj to

$$U : \mathrm{Rng} \longrightarrow \mathrm{Grp} \ ?$$

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