# Lecture 3

## Equalisers & coequalisers

Def$^n$) Let $A \overset{f}{\underset{g}{\rightrightarrows}} B \in \mathcal{C}$.

The equaliser $E$ of $f$ & $g$ comes equipped with an arrow $e : E \to A$ such that the diagram

$$E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B \quad \text{commutes} \quad \left(\begin{matrix}\text{i.e.} \\ fe = ge\end{matrix}\right)$$

& has the universal property that – if $X \xrightarrow{h} A \overset{f}{\underset{g}{\rightrightarrows}} B$ commutes, then $\exists!\ X \xrightarrow{\bar{h}} E$ such that

$$\begin{tikzcd} X \arrow[d, "\bar{h}"'] \arrow[dr, "h"] & \\ E \arrow[r, "e"'] \arrow[ur, phantom, "="] & A \end{tikzcd}$$

- As usual, equalisers are unique up to isomorphism.

Ex) In Set, the equaliser is subset below:

$$\begin{tikzcd} E = \{x \in A : fx = gx\} \arrow[r, hook] \arrow[dr, phantom, "\parallel"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ X \arrow[u, "\bar{h}"] \arrow[ur, "h"'] & & \end{tikzcd}$$

If $fh = gh$ $\Rightarrow f(hx) = g(hx)$ all $x$, so $hx \in E \Rightarrow$ obtain $!\bar{h}$

Ex) In the other algebraic categories such as Grp, Ring etc, equalisers are constructed as in Set:

eg. suppose $f, g : A \rightrightarrows B$ are gp. hom's, form $E \hookrightarrow A$ as above and show $E$ is subgroup:

let $a \in E, b \in E$. Show $a.b \in E$.

$$f(a.b) \underset{\text{as } f \text{ hom}}{=} f(a).f(b) \underset{\text{as } a,b \in E}{=} g(a).g(b) \underset{\text{as } g \text{ hom}}{=} g(a.b)$$

$f(e) = e = g(e)$ so unit $e \in E$

$\Rightarrow$ $E$ is a subgroup ....

- In Grp, a special case of this is illuminating.

Consider zero homomorphism

$$\begin{aligned} G &\xrightarrow{0} H \\ x &\longmapsto 0_H \end{aligned}$$
~ unit of $H$

Then, the equaliser of

$$\{x \in G : fx = 0\} \hookrightarrow G \underset{0}{\overset{f}{\rightrightarrows}} H$$

is the Kernel $\operatorname{Ker} f$ of $f$.

- So kernels are special cases of equalisers.

**Coequalisers** in $\mathcal{C}$ are equalisers in $\mathcal{C}^{op}$: in elementary terms, the coequaliser of

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \text{ is an ob. \& morph.}$$

$B \xrightarrow{k} C$ s.t. $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{k} C$ commutes

given $h$ st $hf = hg$ $\curvearrowright$ $h \searrow D$ then $\downarrow \exists! h$

---

Coequalisers capture quotients, & quotients in algebraic categories are hard to describe explicitly, but we will mention some special cases.

**Example**

- In **Set**, given an equiv. relation $E$ on a set $X$, we can view it as a subset $E \subseteq X^2$ of elts $\{(x,y) : xEy\}$ & then we have

$$E \overset{s}{\underset{t}{\rightrightarrows}} X : (x,y) \overset{s}{\underset{t}{\rightrightarrows}} \begin{matrix} x \\ y \end{matrix}$$

The coequaliser

$$E \overset{s}{\underset{t}{\rightrightarrows}} X \xrightarrow{p} C \text{ is a set}$$

$C$ with property that if $xEy$ then $px = py$, and $p$ is the universal such function:

Indeed, it is $X \xrightarrow{f} X/E$ ~ set of equiv. classes

$x \longmapsto [x]_E$

$[x]_E$ S equiv. class of $x$.

Exercise: check remaining details of this.

---

Ex) Given a normal subgroup $H \trianglelefteq G$, the coequaliser of

$$\begin{tikzcd} H \arrow[r, shift left, "i"] \arrow[r, shift right, "0"'] & G \arrow[r] & ? \end{tikzcd}$$

is, by def$^n$, the universal $G \to ?$ sends all elts of $H$ to $0$.

Indeed, it is the quotient

$$G \longrightarrow G/H$$
$$g \longmapsto gH$$

of $G$ by normal subgroup $H$, familiar from group theory.

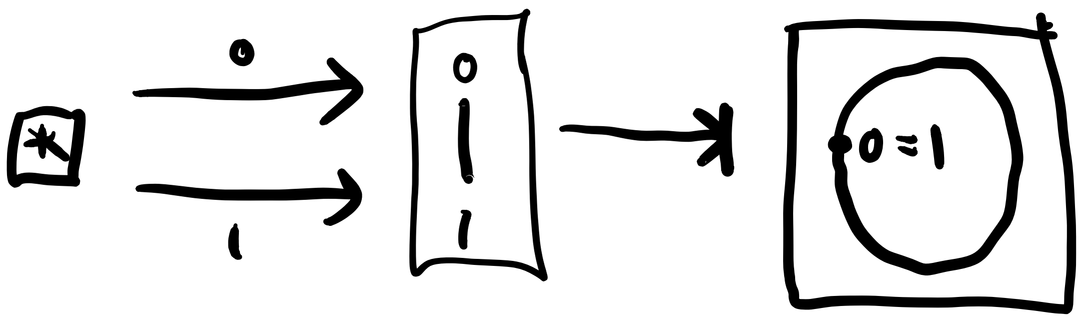
- Coequalisers are also closely rel. to presentations

$$\langle x_1, \ldots, x_n \mid R_1, \ldots, R_n \rangle$$

of algebraic structures via generators & relations.

We will discuss this, & also quotients by congruences (gen. quotients by e-rels & normal subgroups) in the section on universal algebra.

Ex) In topology, coequalisers capture quotient spaces: for instance, the circle is obtained by gluing together the two endpoints of the interval 0——1; it is the coequaliser




Challenge: turn this into a formal proof.

# Limits & colimits in general

- Let $J$ be a small category & $\mathcal{C}$ a category. A Functor $J \longrightarrow \mathcal{C}$ is called a diagram of shape $J$ in $\mathcal{C}$.

**Example**

If $J = \boxed{0 \overset{i}{\underset{j}{\rightrightarrows}} 1}$ a diagram

$J \xrightarrow{D} \mathcal{C}$ is specified by obs & morphisms

$$\begin{array}{ccc} A & \overset{F}{\underset{g}{\rightrightarrows}} & B \\ \| & & \| \\ D0 & \overset{Di}{\underset{Dj}{\rightrightarrows}} & D1 \end{array}$$

**Def^n)** Given a diagram $D : J \longrightarrow \mathcal{C}$, a cone on $D$ is an object $A \in \mathcal{C}$ together with morphisms $A \xrightarrow{f_j} Dj$ For each $j \in J$ such that For all $\alpha : j \longrightarrow k \in J$ the triangle

$$\begin{tikzcd} & Dj \arrow[d, "D\alpha"] \\ A \arrow[ur, "f_j"] \arrow[r, "f_k"'] & Dk \end{tikzcd}$$

commutes.

- A <u>limit</u> of $D$ is a cone
$$(L \xrightarrow{f_j} D_j : j \in J)$$
with the universal property that: given any other cone $(A \xrightarrow{f_j} D_j, j \in J)$ there exists a unique morphism $A \xrightarrow{k} L$ such that

$$\begin{tikzcd} & A \arrow[dl, "k"'] \arrow[dr, "f_j"] & \\ L \arrow[rr, "p_j"'] & & D_j \end{tikzcd}$$

commutes for all $j \in J$.

## Example

For $J = \boxed{0 \rightrightarrows 1}$ a diagram $J \xrightarrow{D} \mathcal{C}$ is a pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ & a cone on $D$ consists of an ob. $X$ & maps

$$\begin{tikzcd} & A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ X \arrow[ur, "p_0"] \arrow[r, "p_1"'] & B \end{tikzcd}$$

such that $\underbrace{f \circ p_0 = p_1 = g \circ p_0}$

two cone equations at $0 \rightrightarrows 1$.

- Or, equivalently, a single morphism $X \xrightarrow{p_0} A$ such that $f \circ p_0 = g \circ p_0$ (since then, $p_1$ is composite $fp_0 = gp_0$)

- In this way, we see that the limit of $D$ is precisely the equaliser of $f$ & $g$.

- i.e. limits of shape $J = \boxed{0 \rightrightarrows 1}$ are precisely equalisers.

**Example** Products are limits of shape $\boxed{0 \quad\quad 1}$

**Example** Terminal objects are limits of shape $\boxed{\quad\quad}$

empty.

**Remark** The colimit of a diagram $D : J \longrightarrow \mathcal{C}$ is a cocone

$$\begin{tikzcd} Dj \arrow[d, "D\alpha"'] \arrow[dr, "f_j"] \arrow[d, phantom, shift left=4, "="] & \\ Dk \arrow[r, "f_k"'] & X \end{tikzcd}$$

with the opposite universal property;

equivalently, the limit of $D^{op} : J^{op} \to \mathcal{C}^{op}$.

- Details are left to reader.

**Theorem** Given $D : J \to \mathcal{C}$,
its limit $(\lim D \xrightarrow{p_i} D_j : j \in J)$
is <u>unique</u> up to <u>unique isomorphism</u>.

**Proof**

- Consider limit cones $(A, p_i : A \to D_i)$ & $(B, q_i : B \to D_i)$.

- Since $(B,q)$ is limit cone, $\exists !\ A \xrightarrow{f} B$ s.t.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "p_i"] \arrow[d, phantom, "="] & \\ B \arrow[rr, "q_i"] & & D_i \end{tikzcd}$$

For all $i \in J$.

- Since $(A,p)$ is lim. cone, $\exists !\ B \xrightarrow{g} A$ st

$$\begin{tikzcd} & B \arrow[dl, "g"'] \arrow[dr, "q_i"] \arrow[d, phantom, "="] & \\ A \arrow[rr, "p_i"] & & D_i \end{tikzcd}$$

all $i \in J$.

- Consider $A \underset{g \circ f}{\overset{1_A}{\rightrightarrows}} A$. By the univ. prop. of $(A,p)$, these will be equal if $p_i \circ g \circ f = 1_A \circ p_i$ all $i$.

- But lhs $= q_i \circ f = p_i$ $=$

  so $1_A = g \circ f$.

- Similarly, $f \circ g = 1_B$.

  So $f : A \longrightarrow B$ is an iso, unique in commuting w/ cones in sense

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "p_i"'] & & B \arrow[dl, "q_i"] \\
& D_i &
\end{tikzcd}
$$

all $i$.

$\square$

- Sim., colimits are unique up to unique isomorphism.

## Infinite products

- Given a set $X$, we can view it as a discrete category: all arrows are identities.

Then a diagram $A : X \longrightarrow \mathcal{C}$ consists of a family of objects $(A_x : x \in X)$ & its limit $\prod_{x \in X} A_x$ is the ($X$-indexed) product of the family $(A_x : x \in X)$.

This comes with maps
$$\prod_{x \in X} A_x \xrightarrow{p^x} A_x \text{ for all } x \in X$$
& it is the universal object equipped with such morphisms.

---

Ex: when $\mathcal{C} = \mathrm{Set}$,
$$\prod_{x \in X} A_x = \{ (a_x \in A_x)_{x \in X} \}$$
the infinite cartesian product.

Similarly, for all algebraic categories

---

As special cases,
$X = \boxed{0 \quad 1}$ we obtain ordinary products & when $X = \boxed{\phantom{0}}$ we obtain terminal objects.

**Def$^n$)** A category $\mathcal{C}$ is *complete* if it has limits of all diagrams, & *cocomplete* if it has colimits of all small diagrams.

**Remark)** All algebraic cats - Set, Grp etc are complete & cocomplete.

- It is straightforward to see they are complete; more complicated to prove that they are cocomplete.
- Will not prove these claims in this course.