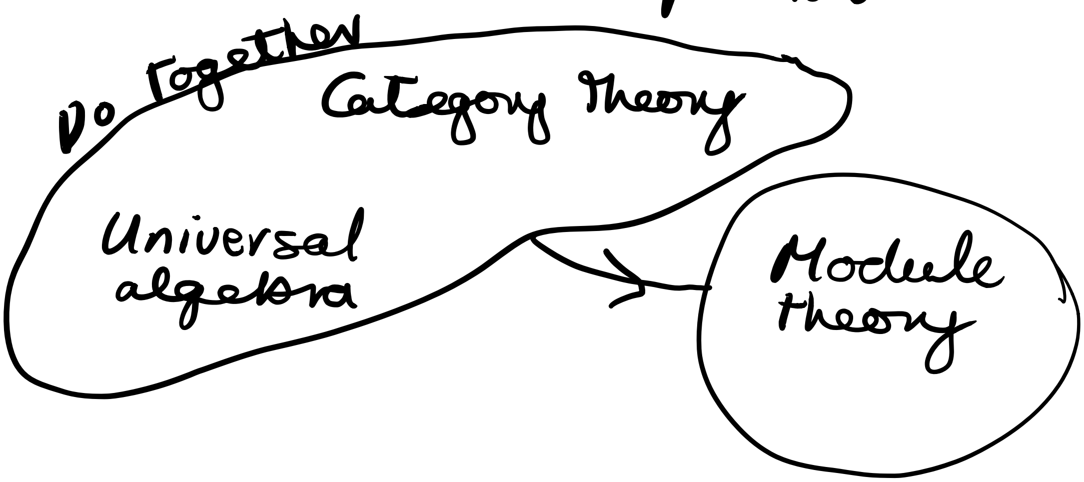
# Algebra 3

- John Bourke - bourkej@math.muni.cz
- Video lectures, slides → IS (weekly)
- Exercise classes Wednesday 10am on Zoom (attendance compulsory)
- Course has 3 components:

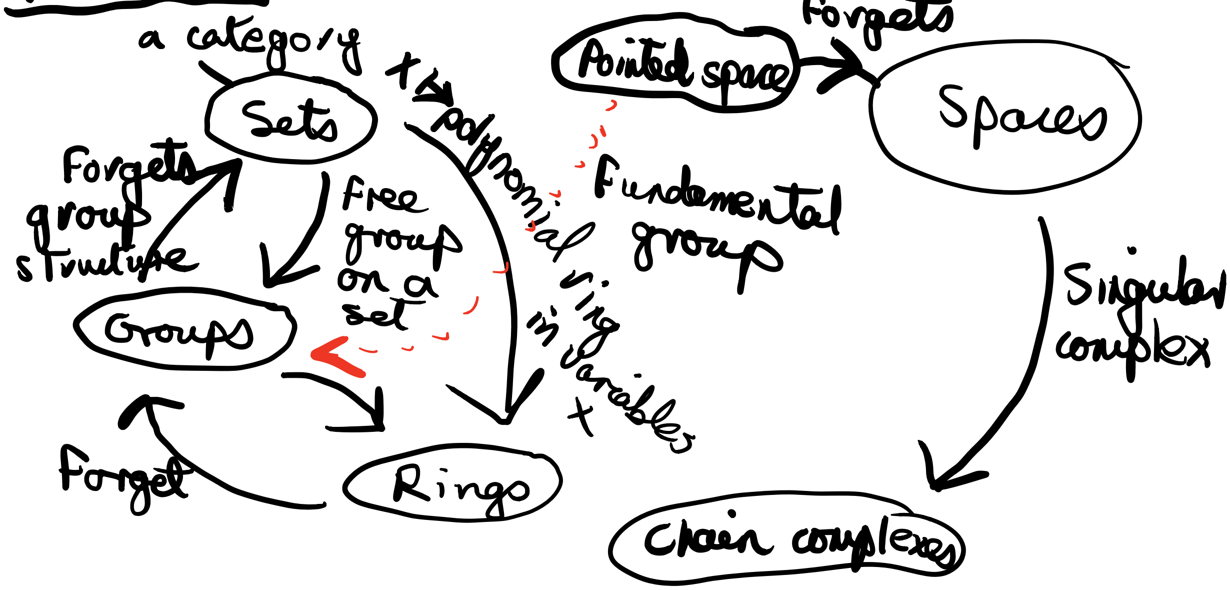



Today: start with category theory.

# What is category theory?

- In math, study structures like sets, groups, rings, topological spaces.
- Each forms category of "structures of that type".
- Cat. theory studies the rel$^n$ between these diff. areas of mathematics (or categories of mathematics)

**Picture**




Category theory :- studies relationships between diff. areas (categories)
- What do they have in common?
- Fundamental notion in category theory is an arrow/morphism:

$$A \longrightarrow B$$

captures relationship between two things.

" Linguistics of mathematics "

Def) A category $\mathcal{C}$ consists of a collection of objects,
& for each pair of objects $A, B \in \mathrm{ob}\,\mathcal{C}$ a collection $\mathcal{C}(A,B)$ of "arrows/morphisms" from $A$ to $B$,

- For each $A, B, C \in \mathrm{ob}\,\mathcal{C}$ a function

$$\begin{aligned} \mathcal{C}(B,C) \times \mathcal{C}(A,B) &\longrightarrow \mathcal{C}(A,C) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

called composition.

- For each object $A \in \mathrm{ob}\,\mathcal{C}$ an arrow $1_A \in \mathcal{C}(A,A)$ called the identity on $A$.
- These satisfy the following axioms:

Associativity

Given $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, $h \in \mathcal{C}(C,D)$ we have $(h \circ g) \circ f = h \circ (g \circ f)$

- Left & right unit laws

Given $f \in \mathcal{C}(A,B)$ we have

$$1_B \circ f = f = f \circ 1_A .$$

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Notation: - Write $A \in \mathcal{C}$ to mean $A \in \mathrm{ob}\,\mathcal{C}$

- $f : A \to B$ or $A \xrightarrow{f} B$ means $f \in \mathcal{C}(A,B)$
- $gf$ to mean $g \circ f$

- Note: Associativity & unit laws imply that given

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$$

there is a <u>unique</u> way to compose the arrows $f_n f_{n-1} \ldots f_2 f_1$, indep. of brackets & identities.

Eg: When $n=4$,

$$((f_4 f_3) f_2) f_1 = (f_4 1_{A_3})((f_3 f_2) f_1).$$

- Commutative diagrams:

Given a diagram such as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h"'] & & B \arrow[d, "g"] \\ C \arrow[r, "i"] & D \arrow[r, "j"] & E \end{tikzcd}$$

we say the <u>diagram commutes</u> if the paths agree: in this case, $gf = jih$.

- In a larger diagram, eg.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h"'] \arrow[drr, phantom, "="] & & B \arrow[r, "k"] \arrow[d, "g"] \arrow[dr, phantom, "="] & F \arrow[d, "l"] \\ C \arrow[r, "i"] & D \arrow[r, "j"] & E \arrow[r, "m"'] & G \end{tikzcd}$$

diag. commutes

the commutativity of subdiagrams implies the commutativity of the outer diagram:

This is <u>diagram chasing</u>.

## Examples

- **Set**: objects are sets,
  morphisms $f : A \to B$ are functions.
- Given $A \xrightarrow{f} B$ & $B \xrightarrow{g} C$ the function $gf : A \to C$ is def. by $gf(x) = g(f(x))$.
  We have $1_A(x) = x$.

## Categories of sets with structure

- **Mon**, the cat of monoids & monoid homomorphisms:
  $f : (A, m_A, e_A) \longrightarrow (B, m_B, e_B)$
  $f(e_A) = e_A$,
  $f(m_A(x,y)) = m_B(fx, fy)$
- **Grp**, the category of groups & group homomorphisms.
- **Rng** ~ rings, ring hom...
  $k$-**Vect** ~ $k$-vector spaces, lin. transf.
- These are examples of algebraic categories.

- More generally, given

$$\underbrace{\Omega}_{\text{signature}}, \quad (\Omega, E), \quad \underbrace{E}_{\text{set of equations}}$$

we can consider the cat. $(\Omega, E)$-Alg. This captures all of the above examples. Return to this example later.

- Top is the category of topological spaces and continuous functions.

**Def)** A morphism $f : A \to B \in \mathcal{C}$ is an isomorphism if $\exists\, g : B \to A$ such that $gf = 1_A$ & $fg = 1_B$.

**Remark)** Can express this via the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "1_A"'] & B \arrow[d, "g"] \arrow[dr, "1_B"] & \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

We say that $g$ is the inverse of $f$ and write $g = f^{-1}$.

This is justified by:

**Proposition)** If $f : A \to B$ is an iso, then its inverse is unique.

**Proof)** Suppose $g, h : B \to A$ are inverses to $f$.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "1_B"'] \arrow[drr, bend left=40, "g"] \arrow[drr, bend right=60, "h"'] & A \arrow[d, "f"] \arrow[dr, "1_A"] & \\
& B \arrow[r, "h"'] & A
\end{tikzcd}$$

Therefore $g = h$.

In alg. notation, the corresponding proof is

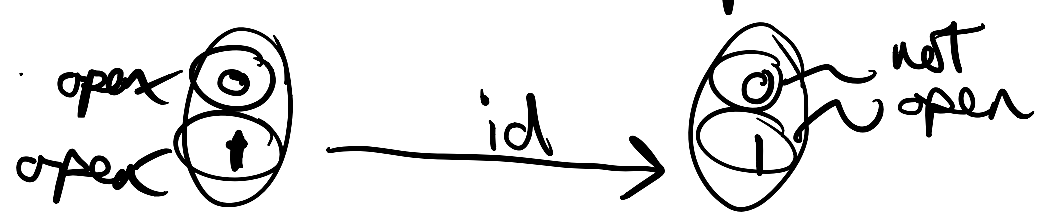
$$g = 1_A \circ g = (h \circ f) \circ g = h \circ (f \circ g)$$
$$= h \circ 1_B = h.$$

## Examples

- In Set, the isomorphisms $f : A \to B$ are the bijections.
- In other algebraic cats such as Grp, an isomorphism is sometimes defined as a bijective homomorphism: this coincides with the categorical notion, since if a homomorphism is bijective its inverse (as a function) is also a homomorphism.
- For Top, an isomorphism is a homeomorphism: a cts function with a cts inverse.

In Top, $\exists$ bijective cts maps which are not homeomorphisms

eg.



not a homeomorphism.

- In summary, the categorical def$^n$ of isomorphism captures the correct notion in all of our examples.

**Def$^n$)** - A category $\mathcal{C}$ is said to be locally small if for all $A, B \in \mathcal{C}$ the collection $\mathcal{C}(A,B)$ is a set.

- If, furthermore, the coll$^n$ ob$\mathcal{C}$ is a set, we say that it is small.

**Examples**

- Set is not small: one cannot form a "set of all sets". However, it is locally small as the collection $\mathrm{Set}(A,B)$ of functions from $A$ to $B$ forms a set.
- All of the examples considered so far are locally small,

but not small.

Here are examples of small categories.

$\boxed{\text{Ex. 1}}$ $\emptyset$ empty cat,

$1 = (\bullet)$ ⇝ 1 object only, id. morphs.

plus identities

$$\boxed{0 \longrightarrow 1}\ , \quad \boxed{\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[ur, "f"] \arrow[r, "h"'] & c \end{tikzcd}}\ , \ \ldots$$

$\boxed{\text{Ex. 2}}$ **Preorders & posets**

- Preorder $(X, \leq)$, satisfying $x \leq x$,
  $x \leq y$ & $y \leq z \Rightarrow x \leq z$

  Poset: also $x \leq y$ & $y \leq x \Rightarrow x = y$.

- If $(X, \leq)$ a preorder, can form a category $X^*$ whose objects are elements of $X$ & such that there exists a single morph. $x \to y \iff x \leq y$ & no morphisms from $x$ to $y$ otherwise.

- In fact,

  Preorders $\overset{\text{same as}}{\equiv}$ small cats with at most 1 morph. between any two objects.

  hopefully later in course (equivalence of cats)

$\boxed{\text{Example 3}}$

If $(\underset{\text{mult}}{M, \times}, \underset{\text{unit}}{e})$ is a monoid we can form a category $\Sigma M$ w/ one object $\bullet$ & whose morphisms $\bullet \xrightarrow{m} \bullet$ are the elements of $M$.

We compose by

$$\begin{tikzcd} \bullet \arrow[r, "m"] \arrow[dr, "n \times m"'] & \bullet \arrow[d, "n"] \arrow[dl, phantom, "="] \\ & \bullet \end{tikzcd} \qquad \bullet \xrightarrow[1]{e} \bullet$$

unit is $1_\bullet$

& assoc. & unit laws for monoid $\Rightarrow$ axioms for a category.

- If $(M, \times, e)$ is a group (all elts. have inverse) then $\Sigma M$ is a groupoid: a category in which all morphisms are isomorphisms.

- $$\begin{array}{ccc} \text{Monoids} & \equiv & \text{1-object small cats} \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \text{Groups} & \equiv & \text{1-ob. small groupoids} \end{array}$$

- **Natural**: e.g. symmetric group $S_n$ consists of the bijections $f \circlearrowright \bar{n} = \{1, \dots, n\}$.

Groups of transformations $\sim$ symmetries of an object.